Gaussian kinetic model for granular gases

James W. Dufty and Aparna Baskaran
Department of Physics, University of Florida, Gainesville, Florida 32611, USA

Lorena Zogaib
Departamento de Matemáticas, Instituto Tecnológico Autónomo de México, México D.F., Mexico
(Received 30 December 2003; published 4 May 2004)

A kinetic model for the Boltzmann equation is proposed and explored as a practical means to investigate the properties of a dilute granular gas. It is shown that all spatially homogeneous initial distributions approach a universal "homogeneous cooling solution" after a few collisions. The homogeneous cooling solution (HCS) is studied in some detail and the exact solution is compared with known results for the hard sphere Boltzmann equation. It is shown that all qualitative features of the HCS, including the nature of overpopulation at large velocities, are reproduced by the kinetic model. It is also shown that all the transport coefficients are in excellent agreement with those from the Boltzmann equation. Also, the model is specialized to one having a velocity independent collision frequency and the resulting HCS and transport coefficients are compared to known results for the Maxwell model. The potential of the model for the study of more complex spatially inhomogeneous states is discussed.

I. INTRODUCTION

Many features of granular gases are captured by an idealized system of smooth, hard spheres with inelastic collisions [1,2]. During the past decade considerable attention has been given to this simple system to understand better the mechanisms behind observed qualitative differences between real gases and those comprised of grains. Among the most productive theoretical tools for analysis used is that of kinetic theory [3], and more specifically at low density, the Boltzmann equation [4,5]. In recent years important conceptual issues, such as the applicability of fluid dynamical equations, have been clarified and quantitative methods have been developed for accurate predictions over a wide range of experimental conditions. It is fair to say that the Boltzmann kinetic theory is now the primary theoretical method for a quantitative description of granular gases.

While there are accurate and efficient numerical algorithms for solving the Boltzmann equation [6], analytic approximations are more limited and exact solutions nonexistent. Such analytic results are of considerable interest because they complement numerical solutions with a more penetrating explication of the dominant mechanisms involved in a specific application. The mathematical complexity of the Boltzmann collision operator is the limiting factor in making progress, so simpler "kinetic models" have been proposed [7]. This approach has been used with great success for real gases with elastic collisions where several exact solutions far from equilibrium have been obtained and shown to be in semiquantitative agreement with the numerical simulations of the Boltzmann equation [8,9]. Recent applications of kinetic models to granular gases have yielded similar interesting exact results [10–12]. The collision operator for a kinetic model is constrained to preserve the most important exact properties of the Boltzmann collision operator, such as a special homogeneous solution and the macroscopic balance equations for mass, momentum, and energy. Otherwise, the model is chosen for simplicity and tractability. The objective here is to recall one of the first kinetic models proposed for granular gases, the Brey-Moreno-Dufty (BMD) model [13], and to generalize it for both a qualitative and a quantitative representation of the Boltzmann equation. It will be referred to as the Gaussian model for reasons that will become apparent.

In the following sections the Gaussian model is defined and applied to the simplest cases of homogeneous states and weak spatial perturbations of those states. The motivation for this work is to provide a tool for a subsequent more detailed study of spatially inhomogeneous states. For example, recent results suggest that the spectrum of the linearized collision operator for a realistic kinetic model could shed important light on the validity conditions for a hydrodynamic description [14,15]. Also, a more practical means to describe boundary value problems is desired for a more faithful comparison with experiments. Attention is focused here on spatially homogeneous states for an isolated system and on transport coefficients for small spatial perturbations, as a means to compare and contrast models. A more detailed application to inhomogeneous states and shear flow will be presented elsewhere.

There is extensive current interest in related Maxwell models. Several exact results have been obtained recently for the homogeneous state of an isolated system using these models [16]. In some mathematical respects the Maxwell models are closer to the hard sphere Boltzmann collision operator than the model studied here. However, its predictions (e.g., homogeneous cooling state distribution, transport coefficients) are quite different from those of the Boltzmann equation as recounted below. In contrast, the Gaussian model is structurally simpler but with the capacity to give a better representation of known results for the Boltzmann equation.

The basic results known for the hard sphere Boltzmann equation are summarized in the following section. The ideas of kinetic modeling and some existing models are reviewed

briefly in Sec. III, and the Gaussian model is defined in Sec. IV. It is an extension and synthesis of two earlier models, the ellipsoidal statistical (ES) model [17] introduced for gases with elastic collisions to yield the correct Prandtl number, and the BMD model noted above for inelastic collisions. The model is constructed to describe both elastic and inelastic collisions, retain a realistic velocity dependent collision frequency, and to yield the correct Prandtl number for accurate transport coefficients.

Also in Sec. IV, the exact solution to the Gaussian model is obtained for an arbitrary initial homogeneous state. It is shown that this class of solutions approaches a universal homogeneous cooling solution (HCS) on a time scale of the order of several collisions. Hence the HCS is the special state for homogeneous granular gases analogous to the Maxwellian for normal gases. The properties of the HCS distribution are shown to be similar to those for the Boltzmann equation at both small and large velocities. In particular, similar exponential decay occurs at large velocities. The special case of a velocity independent collision frequency is studied as well. In that case the HCS is the same as that for the BMD model. The Maxwell models also have a velocity independent collision frequency. In this special case the Gaussian model HCS has properties quite similar to those of the Maxwell models, including algebraic decay for large velocities.

The Chapman-Enskog solution to the Boltzmann equation for small spatial variations around the local HCS applies for the models as well. To Navier-Stokes order this solution is characterized by three transport coefficients. These transport coefficients are compared for the various models and the Boltzmann equation in Sec. V. The Gaussian model with velocity dependent collision frequency provides an excellent representation of the hard sphere Boltzmann results, over a wide range of inelasticity.

The HCS is the reference state for linear hydrodynamics. Consequently, knowledge of the exact form of this distribution function from the kinetic models provides some additional insight there as well. This is illustrated in Sec. VI where the velocity dependence of the hydrodynamic modes [15,14] is calculated. The relationship of these modes to the fluxes in Green-Kubo expressions for transport coefficients is also noted. Finally, the results presented here are summarized in the last section and some other interesting applications of the kinetic model are suggested.

II. HARD SPHERE BOLTZMANN EQUATION

The system considered is composed of N smooth hard spheres of diameter σ in a large volume V. If the density is sufficiently small, $N\sigma^3/V \ll 1$, the one-particle distribution function $f(\mathbf{r},\mathbf{v},t)$ for the number of particles with position $\mathbf{r}$ and velocity $\mathbf{v}$ at time t is determined from the Boltzmann equation [4,5]

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\right)f(\mathbf{r},\mathbf{v},t) = J(\mathbf{r},\mathbf{v}|f(t)). \tag{1}$$

The Boltzmann collision operator J has the form

$$J(\mathbf{r},\mathbf{v}|f(t)) \equiv -\nu(\mathbf{r},t,v)f(\mathbf{r},\mathbf{v},t)$$
$$+ \int d\mathbf{v}_1 \int d\hat{\sigma} K(\hat{\sigma}\cdot\mathbf{g}')\alpha^{-1}f(\mathbf{r},\mathbf{v}',t,)f(\mathbf{r},\mathbf{v}_1',t), \tag{2}$$

where $\mathbf{V}=\mathbf{v}-\mathbf{u}$ and $\mathbf{u}$ is the average flow velocity defined by Eq. (7) below. The first term on the right side represents the loss of particles with velocity $\mathbf{v}$ at a rate due to the collision frequency $\nu(\mathbf{r},V,t)$,

$$\nu(\mathbf{r},V,t) = \int d\mathbf{v}_1 \int d\hat{\sigma} K(\hat{\sigma}\cdot\mathbf{g})f(\mathbf{r},\mathbf{v}_1,t). \tag{3}$$

The second term of Eq. (2) represents the gain of particles with velocity $\mathbf{v}$, where $\{\mathbf{v}',\mathbf{v}_1'\}$ are the "restituting" velocities that lead to $\{\mathbf{v},\mathbf{v}_1\}$ following a smooth, inelastic hard sphere collision:

$$\mathbf{v}' = \mathbf{v} - \tfrac{1}{2}(1+\alpha^{-1})(\hat{\sigma}\cdot\mathbf{g})\hat{\sigma}, \quad \mathbf{v}_1' = \mathbf{v}_1 + \tfrac{1}{2}(1+\alpha^{-1})(\hat{\sigma}\cdot\mathbf{g})\hat{\sigma}. \tag{4}$$

Here, $\hat{\sigma}$ is a unit vector along the line of their centers, and $\mathbf{g}=\mathbf{v}-\mathbf{v}_1$. The parameter α is the coefficient of restitution, $0<\alpha\leqslant 1$, describing the fractional change in the normal component of the relative velocity ($\hat{\sigma}\cdot\mathbf{g}'=-\alpha\hat{\sigma}\cdot\mathbf{g}$) and hence the inelasticity of collisions ($\alpha=1$ corresponds to elastic collisions). The kernel $K(\hat{\sigma}\cdot\mathbf{g})$ is proportional to the flux of particles times the differential cross section and is given by

$$K(\hat{\sigma}\cdot\mathbf{g}) = \sigma^2\Theta(\hat{\sigma}\cdot\mathbf{g})(\hat{\sigma}\cdot\mathbf{g}), \tag{5}$$

where Θ is the Heaviside step function.

The most important properties of the collision operator are those that result from the microscopic balance equations for mass, momentum, and energy in a two-particle collision. For the collision rules (4) it follows directly that J has the following exact properties:

$$\int d\mathbf{v}\begin{pmatrix} 1 \\ \mathbf{v} \\ \tfrac{1}{2}m(\mathbf{v}-\mathbf{u})^2 \end{pmatrix} J(\mathbf{r},\mathbf{v}|f(t)) = \begin{pmatrix} 0 \\ \mathbf{0} \\ -\tfrac{3}{2}nT\zeta \end{pmatrix}, \tag{6}$$

where m is the mass, n is the density, T is the temperature, and $\mathbf{u}$ is the macroscopic flow velocity:

$$\begin{pmatrix} n(\mathbf{r},t) \\ n(\mathbf{r},t)\mathbf{u}(\mathbf{r},t) \\ \tfrac{3}{2}n(\mathbf{r},t)T(\mathbf{r},t) \end{pmatrix} = \int d\mathbf{v}\begin{pmatrix} 1 \\ \mathbf{v} \\ \tfrac{1}{2}m(\mathbf{v}-\mathbf{u})^2 \end{pmatrix} f(\mathbf{r},\mathbf{v},t). \tag{7}$$

The two zeros on the right side of Eq. (6) correspond to conservation of mass and momentum. The last term results from nonconservation of energy and implies the cooling equation for homogeneous states:

$$T^{-1}\partial_t T = -\zeta, \tag{8}$$

where ζ is the "cooling rate,"

$$\zeta = (1-\alpha^2)\frac{m\pi\sigma^2}{24nT}\int d\mathbf{v}\int d\mathbf{v}_1 \, g^3 f(\mathbf{r},\mathbf{v},t,)f(\mathbf{r},\mathbf{v}_1,t). \quad (9)$$

It is easy to verify from Eq. (6) that there is no spatially homogeneous steady state for the isolated system, in contrast to gases with elastic collisions. Instead, there exists a special solution, the HCS, which is assumed to be approached in a few collision times by all homogeneous initial conditions. It has a scaling property such that the dependence on time occurs only through the temperature:

$$f_{\mathrm{HCS}}(\mathbf{v},t) = nv_0^{-3}(t)\phi(v^*), \quad v^* = v/v_0(t), \quad v_0(t) = \sqrt{2T(t)/m}. \quad (10)$$

In the following $v_0(t)$ will be referred to as the thermal velocity in analogy to a gas with elastic collisions. Substitution of Eq. (10) into the Boltzmann equation leads to a time independent equation for $\phi(v^*)$ that must be solved self-consistently with the determination of ζ from Eq. (9),

$$\frac{1}{2}\zeta^*\left(3 + v^*\frac{\partial}{\partial v^*}\right)\phi(v^*) = J^*(v^*|\phi), \quad \zeta^* \equiv \frac{\zeta}{\nu_0}, \quad J^* = \frac{J}{\nu_0}, \quad (11)$$

where ν_0 is an average collision frequency

$$\nu_0(t) = \frac{16}{5}n\sigma^2\sqrt{\frac{\pi}{2}}v_0(t). \quad (12)$$

Because of the scaling property of f_{HCS} both ζ^* and J^* are independent of time. This problem has been studied in detail in recent years and only a few results will be quoted here. For $v^* \lesssim 2$ the solution to Eq. (11) can be expanded in Sonine polynomials about a Maxwellian with the result [18]

$$\phi(v^*) \to \pi^{-3/2}e^{-v^{*2}}\left[1 + \frac{c(\alpha)}{4}\left(v^{*4} - 5v^{*2} + \frac{15}{4}\right)\right], \quad (13)$$

with

$$c(\alpha) \to c_B(\alpha) = \frac{32(1-\alpha)(1-2\alpha^2)}{81 - 17\alpha + 30\alpha^2(1-\alpha)}. \quad (14)$$

The subscript B has been included on $c_B(\alpha)$ here to distinguish the value determined by the hard sphere Boltzmann equation from that for the models introduced below. The accuracy of the representation (13) at small v to within a few percent for all α has been confirmed by Monte Carlo simulation [19,20]. The cooling rate calculated from Eq. (13) is [21]

$$\zeta^* = \frac{5}{12}(1-\alpha^2)\left(1 + \frac{3}{32}c_B(\alpha)\right). \quad (15)$$

For asymptotically large velocities $\phi(v^*)$ has a qualitatively different behavior [18],

$$\phi(v^*) \to Ae^{-(2\beta_B/\zeta^*)v^*}, \quad \beta_B(\alpha) = \frac{5\sqrt{2\pi}}{16}. \quad (16)$$

The constant β_B arises in the large velocity limit of the collision frequency,

$$\nu^*(V^*) \equiv \frac{\nu(v,t)}{\nu_o(t)} \to \beta_B v^*. \quad (17)$$

Thus the origin of the exponential decay is the asymptotic velocity dependence of the collision frequency. The overpopulation in the tail of the distribution, relative to the Gaussian at small velocities, also has been confirmed for $v^* \gtrsim 2$ for all α by Monte Carlo simulation [22,20]. These and the transport properties of Sec. V are the primary results known for the Boltzmann equation with inelastic collisions. They are the main features captured by the kinetic model proposed here.

III. AN OVERVIEW OF KINETIC MODELS

It is remarkable that over a century after Boltzmann wrote his kinetic equation for a low density gas, the content of that equation remains masked by its complexity. Certainly, a great deal is known about solutions near the equilibrium state but the mechanisms controlling nonlinear transport far from equilibrium are still poorly understood. Significant progress has been made in the past 20 years with the development of direct simulation Monte Carlo methods by Bird [6]. This numerical tool is exceptionally powerful and provides access now to a wide range of nonequilibrium states for both elastic and inelastic collisions. For more detailed analytical insight, kinetic models have provided a parallel powerful tool in rarefied gas dynamics. The objective of this section is to give a brief summary of the concept of kinetic models and their extension to inelastic collisions. Although the discussion is limited to the Boltzmann equation, it is noted that the same ideas have been applied as well to its dense fluid generalization, the Enskog kinetic equation, for both fluids and solids [23,24].

A. Maxwell model

The results quoted in the preceding section for the HCS are accurate but not exact. To obtain a more penetrating investigation of this and other solutions a simplified version of the Boltzmann equation called the Maxwell model has been proposed [25] whereby the kernel $K(\hat{\sigma}\cdot\mathbf{g})$ is replaced by a velocity independent kernel $K(\mathbf{r},t)$. Then Eq. (3) implies that the collision frequency also is independent of the velocity, $\nu(\mathbf{r},t) = 4\pi n(\mathbf{r},t)K(\mathbf{r},t)$. The resulting model for the Boltzmann collision operator becomes [26]

$$J(\mathbf{r},\mathbf{v}|f(t)) \to J_M(\mathbf{r},\mathbf{v}|f(t)$$

$$\equiv -\nu(\mathbf{r},t)\left[f(\mathbf{r},\mathbf{v},t) - \frac{1}{4\pi n(\mathbf{r},t)}\int d\mathbf{v}_1\right.$$

$$\left.\times \int d\hat{\sigma}\alpha^{-1}f(\mathbf{r},\mathbf{v}',t,)f(\mathbf{r},\mathbf{v}_1',t)\right]. \quad (18)$$

The collision frequency $\nu(\mathbf{r},t)$ is a free parameter of the model. Its dependence on space and time is due to a possible functional dependence on $f(\mathbf{r},\mathbf{v},t)$. To fix $\nu(\mathbf{r},t)$ the cooling rate is calculated directly for the Maxwell model with the result

$$\zeta_M = \tfrac{1}{6}(1-\alpha^2)\nu(\mathbf{r},t). \qquad (19)$$

The collision frequency is now chosen to assure that the cooling rate for the model is the same as that for the hard sphere Boltzmann equation, $\zeta_M = \zeta$, given to good approximation by Eq. (15). This requires the choice

$$\nu(\mathbf{r},t) = \frac{5}{2}\left(1 + \frac{3}{32}c_B(\alpha)\right)\nu_0(\mathbf{r},t), \qquad (20)$$

where ν_0 is given by Eq. (12). This completely fixes the Maxwell model.

The approximate Maxwell form for the Boltzmann collision operator does not represent any real kinetics due to scattering by a potential. It is called a Maxwell model because the property of K being independent of the velocity follows for scattering by Maxwell molecules interacting via an inverse fourth power law potential. However, the model described here retains the collision rules for inelastic hard spheres, Eq. (4), and therefore is a hybrid not corresponding to any potential. Still, it provides an interesting and tractable model for which several exact results have been obtained recently.

The HCS has been studied for this Maxwell model as well. For small velocities $\phi(v^*)$ again has the form of Eq. (13) except that the coefficient $c(\alpha)$ is replaced by [26]

$$c(\alpha) \to c_M(\alpha) = \frac{12(1-\alpha)^2}{5+3\alpha(2-\alpha)}. \qquad (21)$$

This is significantly different from the small velocity dependence of the hard sphere Boltzmann equation, suggesting that the Maxwell model does not reproduce quantitatively the HCS solution for hard spheres. Furthermore, the difference is even qualitative at larger velocities. The exact asymptotic behavior from the Maxwell model is

$$\phi(v^*) \to A v^{*-k(\alpha)}. \qquad (22)$$

Thus there is algebraic decay for the Maxwell model in contrast to the exponential decay for hard spheres. The exponent $k(\alpha)$ is the solution to a transcendental equation [27,28]

$$1 + \frac{1}{12}(1-\alpha^2)(3-k) = \left(\frac{1+\alpha}{2}\right)^{k-3}\frac{\Gamma\!\left(\dfrac{k-2}{2}\right)}{2\Gamma\!\left(\dfrac{k}{2}\right)} + \int_0^1 dx$$

$$\times\left[1 - \left(1 - \frac{1}{4}(1-\alpha)^2\right)x^2\right]^{(k-3)/2}. \qquad (23)$$

The behavior of $c_M(\alpha)$ and $k(\alpha)$ is illustrated in the following section.

The transport coefficients associated with Navier-Stokes hydrodynamics also have been calculated for the Maxwell model [26]. The agreement with those from the Boltzmann equation for hard spheres is only qualitative (see Sec. V). While the Maxwell model allows interesting and exact solutions, it does not appear to provide a reliable representation of the Boltzmann equation for hard spheres and therefore the results obtained from it must be interpreted with some care.

B. Other kinetic models

The Maxwell model, while simpler than the Boltzmann equation, is still quite complex and even for the HCS the exact distribution function has been calculated only in one dimension. Historically, for normal gases, a number of simpler kinetic models have been applied with great success. More recently, these models have been extended to granular gases with a similar success in applications. To explain them generically, it is useful to rewrite the Boltzmann equation (1) to make the effects of cooling explicit [29]:

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\right)f(\mathbf{r},\mathbf{v},t) - \frac{1}{2}\zeta\,\boldsymbol{\nabla}_{\mathbf{v}}\cdot(\mathbf{V}f) = J'(\mathbf{r},\mathbf{v}|f(t)),$$
$$(24)$$

with

$$J'(\mathbf{r},\mathbf{v}|f(t)) = J(\mathbf{r},\mathbf{v}|f(t)) - \tfrac{1}{2}\zeta\,\boldsymbol{\nabla}_{\mathbf{v}}\cdot(\mathbf{V}f), \qquad (25)$$

where $\mathbf{V} = \mathbf{v} - \mathbf{u}$ is the velocity relative to the average flow. Then the condition (6) becomes

$$\int d\mathbf{v}\begin{pmatrix}1\\ \mathbf{v}\\ \dfrac{1}{2}mV^2\end{pmatrix}J'(\mathbf{r},\mathbf{v}|f(t)) = \begin{pmatrix}0\\ \mathbf{0}\\ 0\end{pmatrix}. \qquad (26)$$

In addition, there is a null space for $J'(\mathbf{r},\mathbf{v}|f(t))$

$$J'(\mathbf{r},\mathbf{v}|f_0(t)) = 0. \qquad (27)$$

The conditions (26) and (27) are the same as those for the conservation laws and the equilibrium state, respectively, for elastic collisions. More generally, Eq. (27) defines the HCS in agreement with Eq. (11). These two sets of conditions are necessary for the macroscopic balance equations (precursors to hydrodynamics) and the "universal" homogeneous state f_0. The basic idea of kinetic models is to replace the actual Boltzmann collision operator by a simpler structure, while preserving the properties (26) and (27).

There are many ways that a kinetic model can be constructed with these constraints. Perhaps the simplest are the BGK model(s) [7]:

$$J'(\mathbf{r},\mathbf{v}|f_0(t)) \to -\nu(\mathbf{r},t)[f(\mathbf{r},\mathbf{v},t) - f_0(\mathbf{r},\mathbf{v},t)]. \qquad (28)$$

Clearly, Eq. (27) is satisfied and the conditions (26) are imposed by requiring that the relevant moments of f and f_0 should be the same:

$$\int d\mathbf{v}\begin{pmatrix} 1 \\ \mathbf{v} \\ \frac{1}{2}mV^2 \end{pmatrix}[f(\mathbf{r},\mathbf{v},t) - f_0(\mathbf{r},\mathbf{v},t)] = \begin{pmatrix} 0 \\ \mathbf{0} \\ 0 \end{pmatrix}. \quad (29)$$

This implies that f_0 is a functional of f so the apparent simplicity of Eq. (28) is misleading. For elastic collisions f_0 is taken to be the local Maxwellian for consistency with the known equilibrium state. In the case of inelastic collisions, it would seem appropriate to choose f_0 as the HCS distribution from the Boltzmann equation. However, since this is not known it is more common to choose again f_0 as the local Maxwellian. As described in the preceding section, this is a reasonable first approximation to the HCS if the velocities are not too large. However, it precludes use of the kinetic model to study the HCS itself. The collision frequency $\nu(\mathbf{r},t)$ is a free parameter of the model, usually chosen to fit one of the transport coefficients. On dimensional grounds $\nu(\mathbf{r},t) \propto n(\mathbf{r},t)T^{1/2}(\mathbf{r},t)$ and therefore also a functional of f.

The Chapman-Enskog solution to the BGK kinetic equation for inelastic collisions has been obtained to derive the associated hydrodynamic equations to Navier-Stokes order (see Sec. V) [21]. The dependence of all transport coefficients on the restitution coefficient α is in good semiquantitative agreement with that for the Boltzmann equation. However, the model suffers from the same well-known problem for elastic collisions of an incorrect Prandtl number $\eta C_p/\kappa$, where $C_p = 5k_B/2m$ is the specific heat per unit mass, η is the shear viscosity, and κ is the thermal conductivity. Since the BGK model has only one parameter ν the absolute value of either the shear viscosity or the thermal conductivity is wrong by a factor of $\approx 2/3$. This can be corrected by choosing for f_0 a more general Gaussian, with an additional parameter leading to the ES model for elastic collisions [17]. BGK models of this type for granular gases have been discussed recently by Astillero and Santos [30].

A related but different kinetic model attempts to represent more directly the gain term of the Boltzmann collision operator [13]. Equation (2) is written as

$$J'(\mathbf{r},\mathbf{v}|f(t)) \equiv -\nu(\mathbf{r},V,t)[f(\mathbf{r},\mathbf{v},t) - g(\mathbf{r},\mathbf{V},t|f)]$$
$$-\tfrac{1}{2}\zeta \boldsymbol{\nabla}_{\mathbf{v}} \cdot (\mathbf{V}f). \quad (30)$$

The gain functional $g(\mathbf{r},\mathbf{V},t|f)$ is now chosen for convenience and simplicity to define the model, but restricted by the exact conditions (26) and (27).

In contrast to the BGK model, the condition (27) now provides an equation that determines a nontrivial HCS solution. The simplest choice for $g(\mathbf{r},\mathbf{V},t|f)$ is again a Maxwellian, but with the temperature modified to account for the extra term on the right side of Eq. (30). For simplicity, applications of this kinetic model to date have also chosen a velocity independent collision frequency. In the limit of elastic collisions it reduces to the BGK model.

The HCS solution can be obtained exactly for this model and, like the Maxwell model, it has algebraic rather than exponential decay at large velocities. The transport coefficients for this second model are of comparable accuracy to

those from the BGK model and suffer from the same difficulty of an incorrect Prandtl number. In the following section the kinetic model based on Eq. (30) is generalized to include a velocity dependent collision frequency and a Gaussian form for $g(\mathbf{r},\mathbf{V},t|f)$ that can accommodate the correct Prandtl number.

IV. GAUSSIAN KINETIC MODEL

The main objective of the present work is to propose a synthesis of the BMD and ES models and to extend them to include a velocity dependent collision frequency. This will be referred to as the Gaussian model. Like the Maxwell model, the Gaussian model admits exact analysis in many interesting cases, but it is simpler and captures more accurately the qualitative features of the Boltzmann equation. In this section the model is defined and the initial value problem is solved exactly for spatially homogeneous states. It is shown that all initial states rapidly approach a universal HCS. The HCS is then studied and compared with known results for the Boltzmann equation for hard spheres. Finally, it is specialized to the case of a velocity independent collision frequency, for comparison with the HCS for the Maxwell model.

The model is defined by the choice of a Gaussian for $g(\mathbf{r},\mathbf{v},t|f)$ in Eqs. (30) [31],

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\right)f(\mathbf{r},\mathbf{v},t) = -\nu(\mathbf{r},V,t)[f(\mathbf{r},\mathbf{v},t) - g(\mathbf{r},\mathbf{V},t|f)],$$

$$(31)$$

$$g(\mathbf{r},\mathbf{V},t|f) = A(\mathbf{r},t)e^{-V_i B_{ij}^{-1}(\mathbf{r},t)V_j}. \quad (32)$$

The scalar function $A(\mathbf{r},t)$ and symmetric tensor $B_{ij}(\mathbf{r},t)$ are determined in part by the conditions (7),

$$\int d\mathbf{v}\begin{pmatrix} 1 \\ \mathbf{v} \\ \frac{1}{2}mV^2 \end{pmatrix}\nu(\mathbf{r},V,t)g(\mathbf{r},\mathbf{V},t|f) = \begin{pmatrix} M_1 \\ \mathbf{M}_2 \\ M_3 - \frac{3}{2}nT\zeta \end{pmatrix},$$

$$(33)$$

where M_i are moments of the distribution function, weighted by the collision frequency,

$$\begin{pmatrix} M_1 \\ \mathbf{M}_2 \\ M_3 \end{pmatrix} = \int d\mathbf{v}\begin{pmatrix} 1 \\ \mathbf{v} \\ \frac{1}{2}mV^2 \end{pmatrix}\nu(\mathbf{r},V,t)f(\mathbf{r},\mathbf{v},t). \quad (34)$$

The choice of a Gaussian is primarily for convenience and simplicity. However, it can be understood also as the result from information theory to determine a function when only the moments in Eq. (33) are specified (see Appendix A). It follows directly that

$$\mathbf{M}_2(\mathbf{r},t) = M_1(\mathbf{r},t)\mathbf{u}(r,t). \tag{35}$$

For the special case of constant collision frequency these become

$$A \to n(\det \pi B)^{-1/2}, \quad \mathbf{M}_2 \to M_1\mathbf{u}, \quad \frac{1}{3}\mathrm{Tr}B \to \frac{T}{m}, \tag{36}$$

so the coefficients of the Gaussian are related to the density, temperature, and flow velocity. This also illustrates that the elements of B are not fully determined by the moment conditions (33).

It is useful to divide the matrix B_{ij} into a part proportional to the unit matrix plus a traceless part:

$$B_{ij} = B\delta_{ij} + \widetilde{B}_{ij}, \quad \widetilde{B}_{ij} = (B_{ij} - B\delta_{ij}), \quad B = \tfrac{1}{3}B_{kk}. \tag{37}$$

In the following a tilde above a matrix will be used to denote its traceless part. The special case of Eq. (36) shows that the trace of B_{ij} is proportional to the temperature and therefore a linear functional of f. It is reasonable to choose the remaining elements of B_{ij} also to have a linear relationship to f. Furthermore, it is required that this traceless part should vanish at f_0, the solution to Eq. (27), which should be isotropic (to agree with the Boltzmann equation),

$$\widetilde{B}_{ij} = \int d\mathbf{v} \left. \frac{\delta \widetilde{B}_{ij}}{\delta f(\mathbf{v})} \right|_{f=f_0} (f - f_0). \tag{38}$$

Thus, the gain $g(\mathbf{r},\mathbf{V},t|f)$ is anisotropic only when evaluated for anisotropic states f. This is an implicit definition since f_0 is a function of B_{ij}. However, the form is such that B_{ij} becomes diagonal when evaluated for $f=f_0$, and hence so does B_{ij}^{-1}. This assures that f_0, when it exists, is isotropic. The trace of B_{ij} is a scalar moment of degree 2 (including the weight factor ν) when $\widetilde{B}_{ij}=0$. Consequently, it is suggestive to take $\widetilde{B}_{ij}$ as the traceless part of the moment of degree 2 of $f-f_0$. The final form for B_{ij} is then

$$B_{ij} = B\delta_{ij} + \frac{y(\alpha)}{nm} \int d\mathbf{v} D_{ij}(\mathbf{V})(f - f_0),$$

$$D_{ij}(\mathbf{V}) = m\left(V_i V_j - \frac{1}{3}\delta_{ij}V^2\right), \tag{39}$$

where $y(\alpha)$ is an undetermined dimensionless quantity independent of the velocity. The conditions (33) and (39) completely determine the parameters A, B_{ij}.

It remains to choose the collision frequency $\nu(\mathbf{r},V,t)$ and the cooling rate $\zeta(\mathbf{r},t)$. In principle, these are specific functionals of f in the Boltzmann equation. Here, they are taken to depend on f only through the temperature and density. The cooling rate is chosen to be the same as the Boltzmann result (15),

$$\zeta(\mathbf{r},t) = \frac{5}{12}(1 - \alpha^2)\left(1 + \frac{3}{32}c_B(\alpha)\right)\nu_0(\mathbf{r},t), \tag{40}$$

$$\nu_0(\mathbf{r},t) = \frac{16}{5}n(\mathbf{r},t)\sigma^2\sqrt{\frac{\pi T(\mathbf{r},t)}{m}}. \tag{41}$$

Similarly, guidance for the choice of the velocity dependent collision frequency $\nu(\mathbf{r},V,t)$ is obtained from that for the Boltzmann [see Eq. (3)]

$$\nu_B(\mathbf{r},V,t) = \pi\sigma^2 \int d\mathbf{V}_1 f(\mathbf{r},\mathbf{V}_1,t)|\mathbf{V} - \mathbf{V}_1|. \tag{42}$$

For small $\mathbf{V}$ this goes to a constant,

$$\nu_B \to \pi\sigma_2 n(\mathbf{r},t)\overline{V}, \quad \overline{V} \equiv \frac{\displaystyle\int d\mathbf{V}_1 f V_1}{\displaystyle\int d\mathbf{V}_1 f}, \tag{43}$$

while for large $\mathbf{V}$ it becomes linear in V,

$$\nu_B \to \pi\sigma 2 n(\mathbf{r},t)V. \tag{44}$$

A representation of the complete velocity dependence for the Gaussian model, preserving these limiting forms, is obtained from Eq. (42) using a Maxwellian for f:

$$\nu(\mathbf{r},V,t) \equiv x(\alpha)\nu_o(\mathbf{r},t)\nu_M^*(V^*), \quad \mathbf{V}^* = \frac{\mathbf{V} - \mathbf{u}}{v_0(t)},$$

$$\nu_M^*(V^*) = \frac{5\sqrt{2}}{16}\left[e^{-v^{*2}} + (2v^* + v^{*-1})\frac{\sqrt{\pi}}{2}\mathrm{erf}\,(v^*)\right]. \tag{45}$$

Here $x(\alpha)$ is a second undetermined dimensionless constant. The particular choice for $x(\alpha)$ and the resulting accuracy of the transport coefficients is discussed in the following section. It is found that $x(\alpha)$ is a smooth function of α of order unity. This form for $\nu^*(v^*)$ has the correct large velocity dependence of Eq. (17) but with the coefficient differing by a factor of $x(\alpha)$ from the Boltzmann equation.

At this point the Gaussian model has been specified in terms of the two remaining constants $x(\alpha)$ and $y(\alpha)$. In the following section it will be shown that the three transport coefficients at Navier-Stokes order are functions of two independent collision integrals. The constants $x(\alpha)$ and $y(\alpha)$ are chosen to assure that these two collision integrals are the same as those from the Boltzmann equation. This leads to a coupled pair of equations [Eqs. (110) and (111) below] that are solved numerically. This completes the definition of the Gaussian model [32].

A. Spatially homogeneous states

In the rest of this section, attention is restricted to spatially homogeneous states. It is shown that for any arbitrary homogeneous initial condition, the solution goes over to a universal HCS in a few collision times. For these initial conditions $\mathbf{u}(\mathbf{r},t)=\mathbf{u}$ is constant and by a Galilean transformation it is possible to choose $\mathbf{u}=\mathbf{0}$. Also from the continuity equation $n(t)=n$ is constant. The temperature obeys the cooling equation (8) which is now written as

$$T^{-1}\partial_s T = -\zeta^*, \quad ds = \nu_0(t)dt, \quad \zeta^* = \frac{\zeta(t)}{\nu_0(t)}. \quad (46)$$

The new time variable s represents the average number of collisions in the time t. It also follows from the definition of $\zeta(t)$ that ζ^* is constant. The temperature therefore has a simple exponential dependence on the collision number:

$$T(s) = e^{-\zeta^* s}T(0). \quad (47)$$

Now consider a general homogeneous initial distribution and look for solutions to the model kinetic equation in the dimensionless form

$$f(\mathbf{v},t) = n v_0^{-3} f^*(\mathbf{v}^*,s), \quad \mathbf{v}^* = \mathbf{v}/v_0(t), \quad \nu^* = \frac{\nu(V,t)}{\nu_o(t)}. \quad (48)$$

The dimensionless form for the model kinetic equation for homogeneous states becomes

$$\left(\partial_s + \frac{1}{2}\zeta^*(3 + \mathbf{v}^* \cdot \boldsymbol{\nabla}_{\mathbf{v}^*}) + \nu^*(V^*)\right)f^*(\mathbf{v}^*,s) = \nu^*(V^*)g^*(\mathbf{v}^*,s|f^*), \quad (49)$$

with

$$g^*(\mathbf{v}^*,s|f^*) = v_0^3 g(\mathbf{v},t|f)/n = A^*(s)e^{-v_i^*(B^{*-1})_{ij}(s)v_j^*}.$$

The moment $\mathbf{M}_2$ vanishes since $\mathbf{u}=\mathbf{0}$. The remaining dimensionless moments are

$$\begin{pmatrix} M_1^*(s) \\ M_3^*(s) \end{pmatrix} = \begin{pmatrix} \dfrac{M_1(t)}{n\nu_0(t)} \\ \dfrac{2M_3(t)}{3nT\nu_0(t)} \end{pmatrix} = \int d\mathbf{v}^* \begin{pmatrix} 1 \\ \dfrac{2}{3}v^{*2} \end{pmatrix} \nu^*(V^*)f^*(\mathbf{v}^*,s). \quad (50)$$

The parameters $A^*(s)$ and $B_{ij}^*(s)$ are related to these by

$$\int d\mathbf{v}^* \begin{pmatrix} 1 \\ \dfrac{2}{3}v^{*2} \end{pmatrix} \nu^*(V^*)A^*(s)e^{-v_i^*(B^{*-1})_{ij}(s)v_j^*} = \begin{pmatrix} M_1^*(s) \\ M_3^*(s)-\zeta^* \end{pmatrix}, \quad (51)$$

$$B_{ij}^* = B^*\delta_{ij} + y\int d\mathbf{v}^*\left(v_i^* v_j^* - \frac{1}{3}\delta_{ij}v^{*2}\right)(f^*-f_0^*). \quad (52)$$

The formal solution to the kinetic equation is found in Appendix B with the result

$$f^*(\mathbf{v}^*,s) = e^{-(3/2)\zeta^* s}K(v^*,s)f^*(e^{-(1/2)\zeta^* s}\mathbf{v}^*,0)$$

$$+ \int_0^s ds' e^{-(3/2)\zeta^* s'}K(v^*,s')\nu^*(e^{-(1/2)\zeta^* s'}v^*)$$

$$\times g^*(e^{-(1/2)\zeta^* s'}\mathbf{v}^*,s-s'), \quad (53)$$

where $K(v^*,s)$ represents the dynamics due to the loss term alone,

$$K(v^*,s) = \exp\left(-\int_0^s ds'' \nu^*(e^{-(1/2)\zeta^* s''}v^*)\right). \quad (54)$$

The collision frequency is a monotonically increasing function of the velocity so $\nu^*(V^*) \geqslant \nu^*(0)$. This gives the inequality

$$K(v^*,s) \leqslant e^{-\nu^*(0)s}. \quad (55)$$

Since $\nu^*(0)$ is of order unity, the domain of integration in Eq. (53) is exponentially bounded for $s>1$ and for large s the integral becomes independent of s. The first term vanishes exponentially fast and the s independent HCS solution is obtained:

$$f^*(\mathbf{v}^*,s) \rightarrow \phi(v^*) = \int_0^\infty ds' e^{-(3/2)\zeta^* s'}K(v^*,s')\nu^*(e^{-(1/2)\zeta^* s'}v^*)$$

$$\times g^*(e^{-(1/2)\zeta^* s'}v^*,\infty). \quad (56)$$

It is readily verified that $\phi(v^*)$ is a stationary solution to Eq. (49) and hence $\phi(v^*)=f_0^*(v^*)$ is the unique HCS solution. It follows from the definition of B that this distribution becomes isotropic on the same time scale as Eq. (54) even if the initial distribution was not:

$$B_{ij}^*(s) \rightarrow \tfrac{1}{3}B_{kk}^*(\infty)\delta_{ij} = B^*\delta_{ij}.$$

For the class of homogeneous initial states considered, this result shows that the HCS is the universal solution after several collisions. Hence it is the special characteristic solution for homogenous states analogous to the Maxwellian for elastic collisions. The result (56) is stronger than the H theorem for elastic collisions in the sense that it implies the approach to the HCS is pointwise in velocity space. It is interesting to observe that this analysis does not require the explicit form for $\nu^*(V^*)g^*(v^*,s|\phi^*)$ and so it applies to models with choices other than the Gaussian. In fact, it applies to the Boltzmann equation itself although in that case Eq. (56) is a more implicit functional relationship whose solution must be proved. The consistency of the moment conditions (50) and (51) is verified in Appendix C using the explicit form (56). The functions $A^*=A^*(\infty)$ and $B^*=B_{kk}^*(\infty)/3$ are fixed by the fact that $\widetilde{B}_{ij}^*=0$ and

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \int d\mathbf{v}^* \begin{pmatrix} 1 \\ \dfrac{2}{3}v^{*2} \end{pmatrix} \phi(v^*). \quad (57)$$

The explicit forms for these equations also are given in Appendix C.

B. The homogeneous cooling state

More explicit properties of the HCS are easily obtained. First, it can be written in the more convenient form
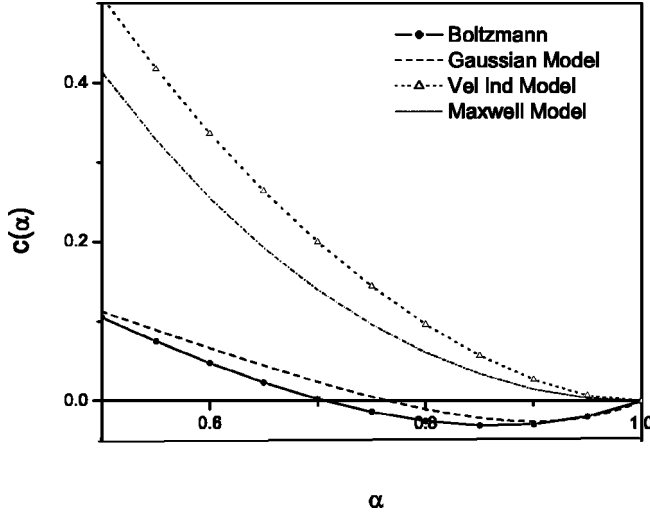
FIG. 1. Comparison of coefficients $c_G(\alpha)$, $c_{G1}(\alpha)$, and $c_M(\alpha)$ with $c_B(\alpha)$.

$$\phi(v^*) = \frac{2A^*}{\zeta^* v^{*3}} \int_0^{v^*} dx x^2 \exp\left(-\frac{2}{\zeta^*} \int_x^{v^*} \frac{dx'}{x'}\right)$$

$$\times \nu^*(x')\Big) \nu^*(x) e^{-B^{*-1}x^2}. \tag{58}$$

The coefficients A^* and B^* are determined from Eq. (57). Both are smooth functions of α with the limiting values $A^* = \pi^{-3/2}$ and $B^* = 1$ at $\alpha = 1$. Practical fits for other values of α in the range 0.4–1 are given by

$$A^* = 0.547 - 0.274\alpha - 0.094\alpha^2,$$

$$B^{*-1} = 1.84 - 0.275\alpha - 0.568\alpha^2.$$

Similarly, fits for $x(\alpha)$ and $y(\alpha)$ are found from Eqs. (110) and (111) to be

$$x(\alpha) = 0.533 + 0.156\alpha - 0.302\alpha^2,$$

$$y(\alpha) = 0.906 - 2.666\alpha + 0.724\alpha^2.$$

For small v^* the form of $\phi(v^*)$ is Gaussian,

$$\phi(v^*) \rightarrow C_1 \exp - C_2 v^{*2}, \tag{59}$$

with

$$C_1 = \frac{A^* p}{3 + p},$$

$$C_2 = \frac{(3+p)}{(5+p)} \left(B^{*-1} - \frac{3}{2(3+p)\nu^*(0)} \frac{d^2\nu^*}{dv^{*2}}\bigg|_{v^*=0} \right), \tag{60}$$

$$p = \frac{2\nu^*(v^*=0)}{\zeta^*}. \tag{61}$$

The coefficient $c_G(\alpha)$ in the polynomial expansion (13) for small v^* is shown for comparison with the corresponding Boltzmann hard sphere result in Fig. 1. A practical fit for c_G in the range of α mentioned above is given by
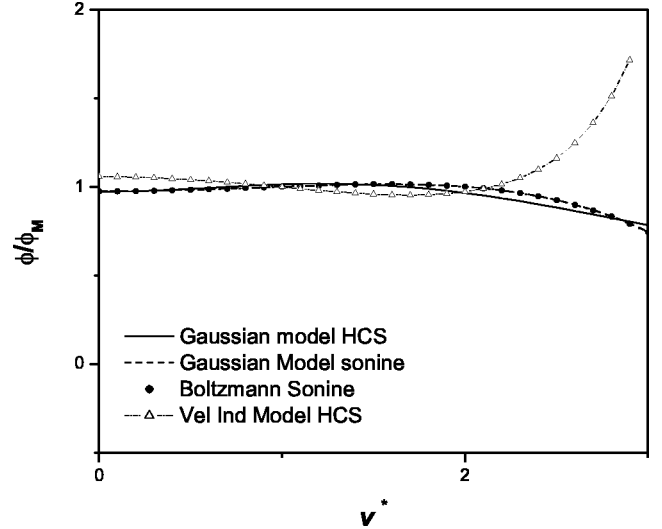

FIG. 2. Comparison of the HCS divided by the Maxwellian for the Gaussian model and the velocity independent collision frequency model with the Sonine approximation using both $c_G(\alpha)$ and $c_B(\alpha)$ for $\alpha = 0.9$.

$$c_G(\alpha) = 0.247 + 0.865\ \alpha - 2.907\ \alpha^2 + 1.793\ \alpha^3.$$

The model is seen to reproduce quite well the Boltzmann results for $\alpha \gtrsim 0.8$ and has the same qualitative behavior for smaller α. As indicated in Fig. 1, the corresponding results for both the Maxwell model and the Gaussian model with velocity independent collision frequency (to be discussed in the following section) are always positive and much larger. This is the first of several observations showing an improvement of the model due to the velocity dependent collision frequency. Figure 2 shows the exact distribution function reduced by the Maxwellian at $\alpha = 0.9$. Also shown are the results from the polynomial expansion using $c_G(\alpha)$ and using $c_B(\alpha)$ for the Boltzmann equation. It is seen that the polynomial expansion follows the exact HCS closely and is close to the expansion using $c_B(\alpha)$. The polynomial expansion using $c_B(\alpha)$ is very close to the actual distribution obtained by Monte Carlo simulation of the Boltzmann equation for $v^* < 2$. Therefore the model gives a good representation of the Boltzmann distribution for $v^* < 2$. This is found to be true over the whole range of α.

The large v^* dependence can be obtained as follows. First, rewrite Eq. (56) as

$$\phi(v^*) = \frac{I(v^*, c)}{v^{*3}} \exp\left(-\frac{2}{\zeta^*} \int_c^{v^*} \frac{dx'}{x'} \nu^*(x')\right), \tag{62}$$

$$I(v^*, c) = \frac{2A^*}{\zeta^*} \int_0^{v^*} dx x^2 \exp\left(-\frac{2}{\zeta^*} \int_x^c \frac{dx'}{x'} \nu^*(x')\right) \nu^*(x) e^{-B^{*-1}x^2}$$

$$= I(\infty, c) - \frac{2A^*}{\zeta^*} \int_{v^*}^{\infty} dx x^2$$

$$\times \exp\left(-\frac{2}{\zeta^*} \int_x^c \frac{dx'}{x'} \nu^*(x')\right) \nu^*(x) e^{-B^{*-1}x^2}. \tag{63}$$

This form holds for any $c \leq v^*$. The integral satisfies the bound

$$I(\infty, c) - I(v^*, c) < I_0 c^2 \nu^*(c) e^{-B^{*-1}c^2}, \qquad (64)$$

where I_0 is a constant. Since B^* is of order unity, this shows that $I(v^*, c) \to I(\infty, c)$ for $v^* \geq c \gg 1$ with deviations of order $\exp(-v^{*2})$. On this scale of velocities the distribution function becomes

$$\phi(v^*) \to \frac{I(\infty, c)}{v^{*3}} \exp\left(-\frac{2}{\zeta^*} \int_c^{v^*} \frac{dx'}{x'} \nu^*(x')\right). \qquad (65)$$

This is not yet the exponential decay quoted in Eq. (16) for the Boltzmann equation. Instead, that exponential decay requires a still larger velocity scale due to the slow approach of $\nu^*(x')$ in Eq. (65) to its large x' limiting form

$$\nu^*(x') \to \beta_G x' + \frac{\beta_G}{2x'} + \text{order } e^{-x'^2}, \qquad (66)$$

where $\beta_G/x(\alpha) = \beta_B$ is the coefficient of the large velocity limit for the collision frequency given in Eq. (17). Thus Eq. (65) behaves as

$$\phi(v^*) \to I(\infty, c) \exp\left[-\left(\frac{2}{\zeta^*}\beta_G(v^* - c) + 3\right) \ln v^* \right.$$
$$\left. + \frac{\beta_G}{\zeta^*}\left(\frac{1}{v^*} - \frac{1}{c}\right)\right]. \qquad (67)$$

The dominant exponential decay is the same as that for the Boltzmann equation, (16), with only the coefficient β_B changed to β_G. In fact, a similar analysis of the derivation of that result for the Boltzmann equation shows that the same intermediate velocity form (67) applies there as well (with $\beta_G \to \beta_B$). The crossover to pure exponential decay requires very large velocities. In practical terms for $\alpha = 0.8$ this form holds to within 0.1% for $v^* \geq 6$ whereas the exponential decay has the same accuracy only for much larger velocities. This is illustrated in Fig. 3 for $\alpha = 0.8$. The derivative of the logarithm of the distribution is plotted so that the initial slope for small velocities is near the Maxwellian value $(-2v)$, while the asymptotic large velocity value is the constant coefficient of the exponential decay $(-2\beta_G/\zeta^*)$ shown as a dotted line. The intermediate crossover is seen to be governed by the asymptotic form (65),

$$v^* \partial_{v^*} \ln \phi(v^*) \sim -\frac{1}{v^*}\left(3 + \frac{2}{\zeta^*}\nu^*(V^*)\right). \qquad (68)$$

Figure 3 also shows that this more general form persists to very large velocities before the final exponential decay is attained. This crossover form is expected to apply for the Boltzmann equation as well and should be taken into account in simulation or experimental attempts to measure the overpopulation at large velocities.

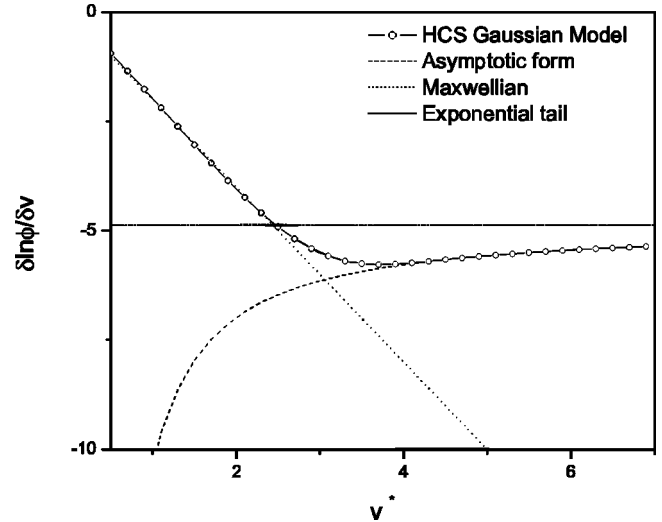To explore the limiting form for $\alpha \to 1$ it is useful to integrate by parts in Eq. (58) to get

FIG. 3. Illustration of the crossover of the HCS for the Gaussian model to the intermediate form, Eq. (67), by plotting $\partial_v \ln \phi(v^*)$ for the HCS, the asymptotic form, and the Maxwellian for $\alpha = 0.8$.

$$\phi(v^*) = \frac{A^*}{v^{*3}} \int_0^{v^*} dx\, x^3 e^{-B^{*-1}x^2} \frac{d}{dx} \exp\left(-\frac{2}{\zeta^*} \int_x^{v^*} \frac{dx'}{x'} \nu^*(x')\right)$$

$$= A^* e^{-B^{*-1}v^{*2}} - \frac{A^*}{v^{*3}} \int_0^{v^*} dx \exp\left(-\frac{2}{\zeta^*} \int_x^{v^*} \frac{dx'}{x'} \nu^*(x')\right)$$

$$\times (3x^2 - 2B^{*-1}x^4) e^{-B^{*-1}x^2}. \qquad (69)$$

The second term on the right side vanishes at $\alpha = 1$ leaving the expected Maxwellian. For $\alpha < 1$ the second term gives the exponential decay at large velocities. In order to dominate the first term it is necessary that $v^* \gg 1$. Since the coefficient in the exponential decay is proportional to $(1-\alpha)^{-1}$ the relevant domain for overpopulation is $v^* > (1-\alpha)^{-1}$. Clearly for $\alpha \to 1$ this overpopulation becomes physically insignificant.

In summary, it has been shown that all of the qualitative features of the HCS for the hard sphere Boltzmann equation are reproduced by the Gaussian model. In the following section it is shown that this quality of the model extends to transport properties as well.

C. Limiting case: Velocity independent collision frequency

To emphasize the effects of the velocity dependence of the collision frequency, it is instructive to consider the same Gaussian model with a velocity independent collision frequency, $\nu^*(V^*) \to \nu^*_{\kappa B}(\alpha)$. The HCS for the Gaussian model then reduces to that of the BMD model. The single parameter of the model, the constant collision frequency, can be chosen to fit the shear viscosity or the thermal conductivity. Due to the choice of $x(\alpha)$ made here the model is tailored to fit the thermal conductivity. The functional form of $\nu^*_{\kappa B}(\alpha)$ is given by Eq. (101) of the following section. The HCS solution (56) simplifies to

$$\phi(v^*) = A^* \int_0^\infty ds e^{-s} e^{-3s/p} \exp \ \ [-B^{*-1} e^{-2s/p} v^{*2}], \quad (70)$$

where $A^*, B^*, M_1^*, M_3^*,$ and p are given by

$$A^* = (\pi B^*)^{-3/2}, \quad B^* = \frac{(p-2)}{p}, \quad M_1^* = M_3^* = \nu^*,$$

$$p = 2\nu_{\kappa B}^*/\zeta^*. \quad (71)$$

A change of variables, $t = e^{-2s/p} v^{*2} p/(p-2)$, allows this to be expressed in terms of an incomplete γ function:

$$\phi(v^*) = \frac{p}{2\pi^{3/2}} \left(\frac{p-2}{p}\right)^{p/2} v^{*-(3+p)} \gamma\left(\frac{p+3}{2}, \frac{p}{p-2} v^{*2}\right), \quad (72)$$

with

$$\gamma(x,y) = \int_0^y dt e^{-t} t^{x-1}. \quad (73)$$

Interestingly, the dimensionless distribution function is entirely characterized by the single constant $p = 2\nu_{\kappa B}^*(\alpha)/\zeta^*(\alpha)$. Its relationship to α is fixed by the choices of cooling rate and collision frequency:

$$p = \frac{8\left[1 + \frac{33}{16}(1-\alpha) + \frac{19-3\alpha}{1024} c_B(\alpha)\right]}{5(1-\alpha)\left[1 + \frac{3}{32} c_B(\alpha)\right]}. \quad (74)$$

For small velocities the representation (13) applies with $c(\alpha)$ given by

$$c(\alpha) \to c_{G1}(\alpha) = \frac{8}{(p-4)p}. \quad (75)$$

Figure 1 shows a comparison of the coefficient $c_{G1}(\alpha)$ with that for the Maxwell model given by Eq. (21). They are seen to be similar for weak dissipation but the Gaussian model grows more rapidly with increasing dissipation. Of course, this difference can be eliminated by a different choice of the parameters for the Gaussian model for a closer agreement to the Maxwell model rather than the hard sphere Boltzmann equation. The accuracy of this polynomial representation is within a few percent for relatively weak dissipation, comparable to that observed for the hard sphere Boltzmann equation. It is clear from this figure that the HCS for models with velocity independent collision frequencies differs from that of the Boltzmann equation and the true Gaussian model at small velocities (see Fig. 2). Also, at small velocities, the HCS in Eq. (70) can be represented as a Gaussian given by

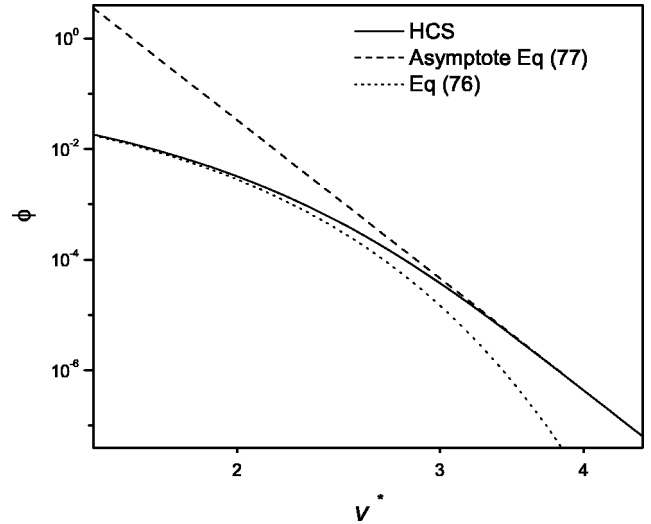$$\phi(v^*) \to \frac{p}{3+p} A e^{-[p(p+3)/(p+5)(p-2)]v^{*2}}. \quad (76)$$

FIG. 4. Illustration of crossover of the HCS for the velocity independent model from the Gaussian given by Eq. (76) to an algebraic decay, Eq. (77) for $\alpha = 0.8$.

The asymptotic behavior for large velocities is obtained from Eq. (72) using the limiting form for the incomplete γ function:

$$\phi(v^*) \to \frac{p}{2\pi^{3/2}} \left(\frac{p}{p-2}\right)^{-p/2} \Gamma\left(\frac{p+3}{2}\right) v^{*-(3+p)}. \quad (77)$$

This algebraic decay is similar to that of the Maxwell model, and in contrast to the exponential decay for the hard sphere Boltzmann equation. This difference is due to neglect of the velocity dependence of the collision frequency in both models. Since the exact solution to the Gaussian model is known the crossover from Gaussian to algebraic forms can be determined explicitly. Figure 4 illustrates this for $\alpha = 0.8$. The crossover domain occurs for $v^* \gtrsim 1$, increasing slightly with decreasing α, with no special intermediate behavior. Figure 5 shows a comparison of the exponent for the algebraic decay for the Gaussian model, $k(\alpha) = 3 + p(\alpha)$, with the corresponding result for the Maxwell model obtained from the solution to Eq. (23). Near $\alpha = 1$ both exponents diverge as $(1-\alpha)^{-1}$ but with different coefficients.

In summary, the simple Gaussian model with constant collision frequency captures semiquantitatively all of the relevant features of the Maxwell model. It has the additional feature of demonstrating explicitly the solution for all homogeneous states to show the rapid transition to the homogeneous cooling state, and the detailed features of that state. The algebraic decay at large velocities implies divergence of moments of degree greater than some critical value for given α. The evolution of such moments for given initial states can be studied in detail to characterize the growing overpopulation of large velocities in the HCS. However, the HCS for both the Maxwell model and the Gaussian model with constant collision frequency differ qualitatively from the Boltzmann result for large and small velocities.
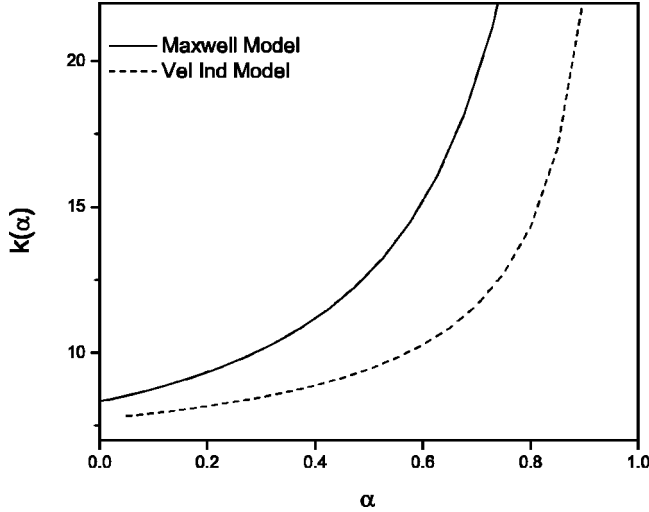
FIG. 5. The exponent of algebraic decay for the Maxwell model, Eq. (23), and velocity independent Gaussian model, Eq. (77) plotted as a function of α.

V. NAVIER-STOKES HYDRODYNAMICS

In this section, states with smooth spatial and temporal variations in the density, temperature, and flow velocity are considered. These are states for which a macroscopic hydrodynamic description is expected to apply. First, the results of the Chapman-Enskog method to solve the kinetic equation is recalled. Next, the transport coefficients obtained from this solution are identified exactly and in a first Sonine polynomial approximation. Finally, these latter expressions are evaluated for the model and compared with the corresponding results for the Boltzmann equation.

A. Chapman-Enskog solution

The hydrodynamic equations for spatially inhomogeneous states are obtained from a special solution to the kinetic equation generated by the Chapman-Enskog method. The method is quite general and requires only the properties (6) for the collision operator. Since these are preserved in the Maxwell and Gaussian models the results obtained earlier for the Boltzmann equation [21] apply for the models as well. The solution is "normal" in the sense that all space and time dependence occurs only through the hydrodynamic fields. To first order in the spatial gradients of these fields it is found to be

$$f(\mathbf{r},\mathbf{V},t) = f^{(0)}(\mathbf{r},\mathbf{V},t) + f^{(1)}(\mathbf{r},\mathbf{V},t), \quad (78)$$

where $\mathbf{V} = \mathbf{v} - \mathbf{u}(\mathbf{r},t)$ is the velocity relative to the flow field. The first term of Eq. (78) is the solution to the kinetic equation to zeroth order in the spatial gradients:

$$\frac{1}{2}\zeta^{(0)}(\mathbf{r},t)\boldsymbol{\nabla}_{\mathbf{V}}\cdot[\mathbf{V}f^{(0)}(\mathbf{r},\mathbf{V},t)] = J(\mathbf{r},v|f^{(0)}(t)), \quad (79)$$

where the superscript on $\zeta^{(0)}$ denotes Eq. (9) evaluated with $f^{(0)}$. Equation (79) is an equation for the velocity dependence of $f^{(0)}(\mathbf{r},\mathbf{V},t)$, which is the *same* as that for the HCS distribution of the previous sections. The dependence on $\mathbf{r},t$ oc-

curs only through the parameters of the HCS. More specifically, $f^{(0)}(\mathbf{r},\mathbf{V},t)$ is the *local* HCS obtained from Eq. (10) by replacing the density, temperature, and flow velocity by their actual values in the spatially inhomogeneous state:

$$f^{(0)}(\mathbf{r},\mathbf{V},t) = n(\mathbf{r},t)v_0^{-3}(\mathbf{r},t)\phi(V^*), \quad V^* = V/v_0(\mathbf{r},t), \quad (80)$$

$$v_0(\mathbf{r},t) = \sqrt{2T(\mathbf{r},t)/m}.$$

The second term on the right side of Eq. (78) is proportional to the gradients,

$$f^{(1)}(\mathbf{r},\mathbf{V},t) = \mathcal{A}\cdot\boldsymbol{\nabla}\ln T + \mathcal{B}\cdot\boldsymbol{\nabla}\ln n$$
$$+ C_{ij}\frac{1}{2}\left(\partial_j u_i + \partial_i u_j - \frac{2}{3}\delta_{ij}\boldsymbol{\nabla}\cdot\mathbf{u}\right). \quad (81)$$

[There are no contributions from the expansion of $\zeta(\mathbf{r},t)$ to first order as this vanishes for both the Boltzmann case and for the models.] The functions $\mathcal{A}(\mathbf{V}|n,\mathbf{u},T)$, $\mathcal{B}(\mathbf{V}|n,\mathbf{u},T)$, and $C_{ij}(\mathbf{V}|n,\mathbf{u},T)$ are solutions to the integral equations

$$\left(-\zeta^{(0)}T\partial_T + \mathcal{L} - \frac{\zeta^{(0)}}{2}\right)\mathcal{A} = \mathbf{A}, \quad (82)$$

$$(-\zeta^{(0)}T\partial_T + \mathcal{L})\mathcal{B} = \mathbf{B} + \zeta^{(0)}\mathcal{A}, \quad (83)$$

$$(-\zeta^{(0)}T\partial_T + \mathcal{L})C_{ij} = C_{ij}, \quad (84)$$

with the definitions

$$\mathbf{A}(\mathbf{V}|n,\mathbf{u},T) = \left(\frac{5}{2} + \frac{1}{2}\mathbf{V}\cdot\boldsymbol{\nabla}_V\right)f^{(0)}\mathbf{V} - \mathbf{V}f^{(0)} - \frac{T}{m}\boldsymbol{\nabla}_V f^{(0)}, \quad (85)$$

$$\mathbf{B}(\mathbf{V}|n,\mathbf{u},T) = -\mathbf{V}f^{(0)} - \frac{T}{m}\boldsymbol{\nabla}_V f^{(0)}, \quad (86)$$

$$C_{ij}(\mathbf{V}|n,\mathbf{u},T) = V_i(\partial_{V_j}f^{(0)}). \quad (87)$$

The linear operator $\mathcal{L}$ is the collision operator expanded to first order in $f^{(1)}$,

$$Lf^{(1)} = \int d\mathbf{v}'\ \frac{\delta J(\mathbf{r},\mathbf{v}|f(t))}{\delta f(\mathbf{r},\mathbf{v}',t)}\bigg|_{f=f^{(0)}}f^{(1)}(\mathbf{r},\mathbf{V}',t). \quad (88)$$

B. Transport coefficients

The Boltzmann equation and all models considered give the same macroscopic balance equations for mass, energy, and momentum (or density, temperature, and flow velocity) because they all imply the properties (6). The Navier-Stokes hydrodynamic equations follow by evaluating the momentum flux P_{ij} and the heat flux q in the macroscopic balance equations using the Chapman-Enskog solution to first order in the spatial gradients, with the results

$$P_{ij} = -\eta\left(\partial_j u_i + \partial_i u_j - \frac{2}{3}\delta_{ij}\nabla\cdot\mathbf{u}\right), \quad \mathbf{q} = -\kappa\nabla T - \mu\nabla n.$$

$$(89)$$

The first of these is Newton's viscosity law, where η is the shear viscosity. The second is a generalization of Fourier's law, where κ is the thermal conductivity. There is an additional contribution for granular gases proportional to the density gradient, with a transport coefficient μ, that does not occur for normal gases. These are identified from the Chapman-Enskog solution as [21]

$$\eta = nT\left(\nu_\eta - \frac{1}{2}\zeta^{(0)}\right)^{-1},$$

$$(90)$$

$$\kappa = \frac{5nT}{2m}(\nu_\kappa - 2\zeta^{(0)})^{-1}(1+c),$$

$$(91)$$

$$\mu = \frac{15T^2}{2m}(2\nu_\mu - 3\zeta^{(0)})^{-1}\left(\zeta^{*(0)}\frac{\kappa}{\kappa_0} + \frac{1}{3}c\right),$$

$$(92)$$

with the definitions

$$\nu_\eta = \frac{\int d\mathbf{V}\, D_{ij}(\mathbf{V})\mathcal{L}\mathcal{C}_{ij}(\mathbf{V})}{\int d\mathbf{V}\, D_{ij}(\mathbf{V})\mathcal{C}_{ij}(\mathbf{V})},$$

$$\nu_\kappa = \frac{\int d\mathbf{V}\, \mathbf{S}(\mathbf{V})\cdot\mathcal{L}\mathcal{A}(\mathbf{V})}{\int d\mathbf{V}\, \mathbf{S}(\mathbf{V})\cdot\mathcal{A}(\mathbf{V})},$$

$$\nu_\mu = \frac{\int d\mathbf{V}\, \mathbf{S}(\mathbf{V})\cdot\mathcal{L}\mathcal{B}(\mathbf{V})}{\int d\mathbf{V}\, \mathbf{S}(\mathbf{V})\cdot\mathcal{B}(\mathbf{V})}.$$

$$(93)$$

The functions $D_{ij}(\mathbf{V})$ and $\mathbf{S}(\mathbf{V})$ are

$$D_{ij}(\mathbf{V}) = m\left(V_iV_j - \frac{1}{3}V^2\delta_{ij}\right), \quad \mathbf{S}(\mathbf{V}) = \mathbf{V}\left(\frac{1}{2}mV^2 - \frac{5}{2}T\right).$$

$$(94)$$

Also, $\kappa_0 = 15\eta_0/4m$ and $\eta_0 = 5(mT)^{1/2}/16\sigma^2\pi^{1/2}$ are the low density values of the thermal conductivity and the shear viscosity in the elastic limit, respectively. The constant $c(\alpha)$ is the same as that occurring in the representation (13), appropriate for either the Boltzmann equation or the model being considered. The forms (90)–(92) provide the exact expressions for these transport coefficients.

C. Sonine polynomial approximation

More explicit results require determination of $f^{(0)}$ and the solutions $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}_{ij}$ to the linear integral equations (82)–(84). The Gaussian model allows explicit construction

of these. However, in general it is useful to represent these quantities as an expansion in a complete set of polynomials and generate approximations by truncating the expansion. In practice the leading term in these expansions provides a very accurate description over the full range of dissipation and density. The determination of $f^{(0)}$ to leading order in the Sonine polynomial has already been given by Eq. (13). Similarly, the leading contributions to the expansions of $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}_{ij}$ are found to be [21]

$$\begin{pmatrix}\mathcal{A}(V)\\ \mathcal{B}(V)\\ \mathcal{C}_{ij}(V)\end{pmatrix} \to f_M(V)\begin{pmatrix}c_\mathcal{A}\mathbf{S}(\mathbf{V})\\ c_\mathcal{B}\mathbf{S}(\mathbf{V})\\ c_\mathcal{C}D_{ij}(\mathbf{V})\end{pmatrix},$$

$$(95)$$

with the coefficients

$$f_M(V) = n(\pi v_0^2)^{-3/2}e^{-(V/v_0)^2},$$

$$\begin{pmatrix}c_\mathcal{A}\\ c_\mathcal{B}\end{pmatrix} = \frac{2m}{15nT^3}\int d\mathbf{V}\begin{pmatrix}\mathcal{A}(V)\cdot\mathbf{S}(\mathbf{V})\\ \mathcal{B}(V)\cdot\mathbf{S}(\mathbf{V})\end{pmatrix} = \begin{pmatrix}-\dfrac{2m}{5nT^2}\kappa\\[2mm] -\dfrac{2m}{5T^3}\mu\end{pmatrix},$$

$$(96)$$

$$c_\mathcal{C} = \frac{T^2}{10n}\int d\mathbf{V}\mathcal{C}_{ij}(V)D_{ij}(\mathbf{V}) = -\frac{T^2}{n}\eta.$$

$$(97)$$

The distribution function $f^{(1)}$ in this approximation is obtained from Eq. (81),

$$f^{(1)} \to -f_M\left[\frac{2m}{5nT^3}(\kappa\nabla T + \mu\nabla n)\cdot\mathbf{S}(\mathbf{V}) + \frac{1}{nT^2}\right.$$
$$\left. \times\eta\frac{1}{2}\left(\partial_i u_j + \partial_j u_i - \frac{2}{3}\delta_{ij}\nabla\cdot\mathbf{u}\right)D_{ij}(\mathbf{V})\right].$$

$$(98)$$

To evaluate the transport coefficients the forms (90)–(92) are used, with the frequencies $\nu_\eta(\alpha)$, $\nu_\kappa(\alpha)$, and $\nu_\mu(\alpha)$ determined in this approximation by

$$\nu_\eta = \frac{\int d\mathbf{V}\, D_{ij}\mathcal{L}f_M D_{ij}}{\int d\mathbf{V}\, f_M D_{ij}D_{ij}}, \quad \nu_\kappa = \nu_\mu = \frac{\int d\mathbf{V}\, \mathbf{S}\cdot\mathcal{L}f_M\mathbf{S}}{\int d\mathbf{V}f_M\,\mathbf{S}\cdot\mathbf{S}}.$$

$$(99)$$

These integrals have been calculated for the Boltzmann equation [21]

$$\nu_{\eta B}^* = \frac{\nu_{\eta B}}{\nu_0(\mathbf{r},t)} = \left(1 - \frac{1}{4}(1-\alpha)^2\right)\left(1 - \frac{1}{64}c_B(\alpha)\right),$$

$$(100)$$

$$\nu^*_{\kappa B} = \nu^*_{\mu B} = \frac{1+\alpha}{3}\left(1 + \frac{33}{16}(1-\alpha) + \frac{19-3\alpha}{1024}c_B(\alpha)\right).$$
(101)

The average local frequency $\nu_0(\mathbf{r},t)$ is given by Eq. (41). The corresponding results for the Maxwell model [26] are

$$\nu^*_{\eta M} = \frac{(1+\alpha)(4-\alpha)}{6}\left(1 + \frac{3}{32}c_B(\alpha)\right),$$
(102)

$$\nu^*_{\kappa M} = \nu^*_{\mu M} = \frac{1}{24}(1+\alpha)(19-11\alpha)\left(1 + \frac{3}{32}c_B(\alpha)\right).$$
(103)

Finally, it is straightforward to perform the same calculations for the Gaussian model considered here. The form of the linearized collision operator $\mathcal{L}$ for the Gaussian model is obtained in Appendix D:

$$\mathcal{L}f^{(1)} = (1-\mathcal{P})\nu f^{(1)} - \frac{5}{2}y\frac{\nu g^{(0)}D_{ij}(\mathbf{V})\int d\mathbf{v}' D_{ij}f^{(1)}}{\int d\mathbf{v} D_{ij}D_{ij}g^{(0)}}.$$
(104)

Here $\mathcal{P}$ is a projection operator defined by

$$\mathcal{P}X = \nu g^{(0)}\psi_\sigma\int d\mathbf{v}\psi_\sigma X, \quad g^{(0)} = g(|f^{(0)}|),$$
(105)

and $\{\psi_\sigma\}$ is the orthonormal set,

$$\begin{pmatrix}\psi_1\\\psi_2\\\psi_3\end{pmatrix} = \begin{pmatrix}\sqrt{\frac{1}{N_1}}1\\\sqrt{\frac{3}{N_2}}\mathbf{V}\\\sqrt{\frac{1}{N_3}}\left(V^2 - \frac{N_2}{N_1}\right)\end{pmatrix}.$$
(106)

The normalization constants N_i are given in Appendix D. With these results the frequencies $\nu_{\eta G}$, $\nu_{\kappa G}$, and $\nu_{\mu G}$ are found to be

$$\nu_{\eta G} = x\left(\frac{\int d\mathbf{V} D_{ij}\nu_M f_M D_{ij}}{\int d\mathbf{V} f_M D_{ij}D_{ij}} - \frac{1}{2}y\frac{\int d\mathbf{V} D_{ij}\nu_M g^{(0)}D_{ij}}{\int d\mathbf{V} D_{ij}g^{(0)}D_{ij}}\right),$$
(107)

$$\nu^*_{\kappa G} = \nu^*_{\mu G} = \frac{x\int d\mathbf{V} \mathbf{S}\cdot(1-\mathcal{P})\nu^*_M f_M\mathbf{S}}{\int d\mathbf{V} f_M S^2}.$$
(108)

The constants $x(\alpha)$ and $y(\alpha)$ are now chosen to assure accurate transport coefficients. This is most directly done by re-

quiring that the above frequencies are the same as those from the Boltzmann equation, i.e.,

$$\nu_{\eta G} = \nu_{\eta B}, \quad \nu_{\kappa G} = \nu_{\kappa B}.$$
(109)

It follows from Eqs. (90) and (91) that the Prandtl number at $\alpha=1$ is ν_κ/ν_η. So this choice assumes that the Gaussian model also will have the correct Prandtl number in the elastic limit. This gives

$$x(\alpha) = \nu^*_{\kappa B}(\alpha)\frac{\int d\mathbf{V} f_M S^2}{\int d\mathbf{V} \mathbf{S}\cdot(1-\mathcal{P})\nu^*_M f_M\mathbf{S}},$$
(110)

$$y(\alpha) = -2\left(\nu_{\eta B}(\alpha) - x(\alpha)\frac{\int d\mathbf{V} D_{ij}\nu_M f_M D_{ij}}{\int d\mathbf{V} f_M D_{ij}D_{ij}}\right)$$

$$\times\left(\frac{x(\alpha)\int d\mathbf{V} D_{ij}\nu_M g^{(0)}D_{ij}}{\int d\mathbf{V} g^{(0)}D_{ij}D_{ij}}\right)^{-1}.$$
(111)

With these choices, the transport coefficients are given by Eqs. (90)–(92), and the only differences from the Boltzmann values results from the replacement of $c_B(\alpha)$ by $c_G(\alpha)$ in the expressions for κ and μ. It should be noted that Eqs. (111) and (110) are implicit since the right sides depend on $x(\alpha)$ through the collision frequency in Eq. (57) that determines the parameters A and B_{ij}. In practice the calculation of $x(\alpha)$ is done iteratively. First, the integrals in Eq. (110) are evaluated at $\alpha=1$ to determine a zeroth-order estimate for $x(\alpha)$. Then Eq. (57) is used to get a first approximation to A and B_{ij}. Next, these results are used in Eqs. (111) and (110) to calculate the first approximation to $x(\alpha)$ and $y(\alpha)$. The process is repeated starting with this first approximation for $x(\alpha)$. The results reported here are for two iterations, showing good convergence of the process. The fits obtained for x and y are

$$x(\alpha) = 0.533 + 0.156\alpha - 0.302\,\alpha^2,$$

$$y(\alpha) = 0.906 - 2.666\alpha + 0.724\alpha^2.$$

For $\alpha=1$ these results reduce to

$$\nu^*_{\eta G}(1) = \nu^*_{\eta B}(1) = 1, \quad \nu^*_{\kappa G}(1) = \nu^*_{\mu G}(1) = \nu^*_{\kappa B}(1) = \frac{2}{3},$$
(112)

$$x(1) = \frac{448}{1153}, \quad y(1) = -\frac{1247}{1106}.$$
(113)

These results define a new kinetic model for normal gases, extending the ES model to one with a more realistic velocity dependent collision frequency.

Also, for the special case of a constant collision frequency the general results reduce to
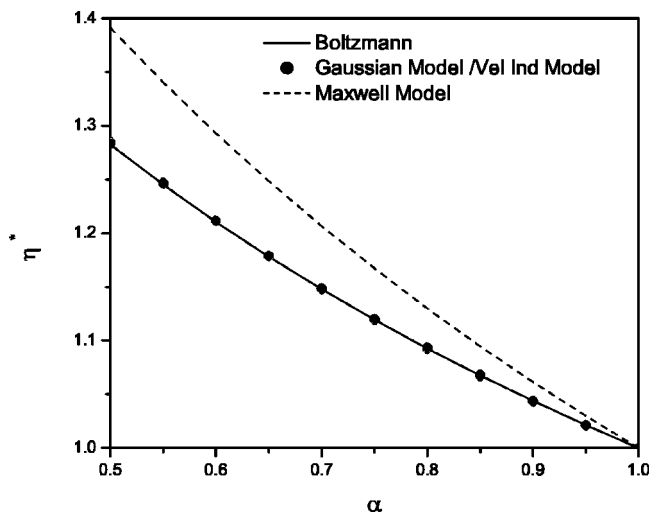
FIG. 6. Plot of the fitted dimensionless viscosity $\eta^* = \eta/\eta_0$ for the velocity dependent and velocity independent collision frequency Gaussian models with the Boltzmann and Maxwell model results.

$$x(\alpha)\nu_M^* = \nu_{\kappa B}^*(\alpha), \quad y(\alpha) = -2\left(\frac{\nu_{\eta B}^*(\alpha)}{\nu_{\kappa B}^*(\alpha)} - 1\right). \quad (114)$$

This case is relevant also if one wanted to use the Gaussian kinetic model to represent the Maxwell model. Then in Eq. (114) $\nu_{\eta B}^*(\alpha)$ and $\nu_{\kappa B}^*(\alpha)$ should be replaced by $\nu_{\eta M}^*(\alpha)$ and $\nu_{\kappa M}^*(\alpha)$, respectively. Finally, for both $\alpha = 1$ and constant collision frequency the usual ellipsoidal statistical model is recovered:

$$\nu_{\eta G}^*(1) = 1, \quad \nu_{\kappa G}^*(1) = \nu_{\mu G}^*(1) = x\nu_M^* = \tfrac{2}{3}, \quad y = -1.$$
$$(115)$$

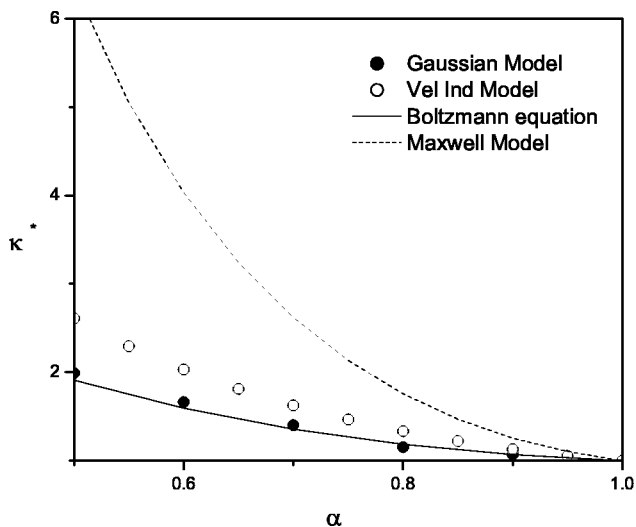Figures 6–8 show the shear viscosity, thermal conductiv-

FIG. 7. Comparison of dimensionless thermal conductivity $\kappa^* = \kappa/\kappa_0$ as calculated from the velocity dependent and velocity independent collision frequency Gaussian models with the Boltzmann and Maxwell model results.
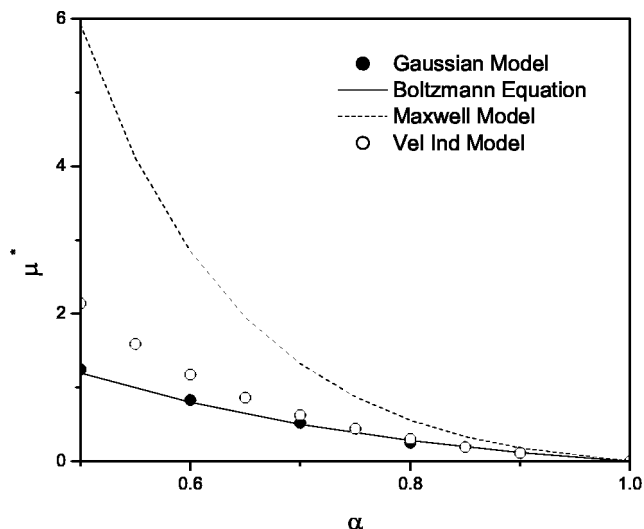

FIG. 8. Comparison of $\mu^* = \mu n/T\kappa_0$ as calculated from the Gaussian models with the Boltzmann and Maxwell model results.

ity, and the new transport coefficient μ for the various models compared with the Boltzmann equation results. It is seen that the shear viscosity for the Gaussian model with either a velocity dependent or velocity independent collision frequency is indistinguishable from the Boltzmann result. The small differences between the velocity dependent collision frequency Gaussian model and the Boltzmann results for the κ and μ coefficients are due to the differences between $c_B(\alpha)$ and $c_G(\alpha)$, and by truncation of the above iteration solution for $x(\alpha)$ after two steps. The differences in the case of the constant collision frequency models are more pronounced because as seen in Fig. 1, $c_{G1}(\alpha)$ and $c_M(\alpha)$ are significantly different from $c_B(\alpha)$ for smaller α values. These results show that the Gaussian model has the ability to fit the transport properties to the hard sphere Boltzmann results for all α, including the correct Prandtl number $\eta C_P/\kappa = 2/3$ at $\alpha = 1$. The other model do not have this capacity and the associated transport coefficients do not represent as well those from the Boltzmann equation [26], although they yield the correct Prandtl number at $\alpha = 1$. Clearly, the inclusion of the velocity dependent collision frequency in the model allows excellent agreement with the Boltzmann results.

VI. HYDRODYNAMIC MODES AND GREEN-KUBO EXPRESSIONS

The simplest solutions to the Navier-Stokes equations are those for a large system with small perturbations about the HCS (not the local HCS as considered above). The resulting five independent solutions are referred to as hydrodynamic modes. For a gas with elastic collisions, these would correspond to shear diffusion, heat diffusion, and damped sound propagation. The hydrodynamic modes are more complicated for inelastic collisions but their properties have been worked out and discussed [21].

The Chapman-Enskog method provides a normal solution that implicitly presumes the existence of a hydrodynamic description. A more fundamental study of the context or va-

lidity of hydrodynamics is possible by determining the possible solutions to the Boltzmann equation for small perturbations of the HCS. The resulting linearized Boltzmann equation is obtained by substituting $f = f_{HCS}[1 + \Delta]$ into Eq. (1) and retaining terms linear in Δ,

$$(\partial_s + \mathbf{v}^* \cdot \boldsymbol{\nabla}^* + \mathcal{L}_0)\Delta = 0. \qquad (116)$$

The dimensionless units of Sec. IV have been used, and the linear operator $\mathcal{L}_0$ is defined by

$$\mathcal{L}_0 \Delta = \phi^{-1} \mathcal{L}(\phi \Delta) - \phi^{-1} \frac{\zeta_0}{2} \frac{\partial}{\partial \mathbf{v}^*}(\phi \Delta). \qquad (117)$$

For elastic collisions the second term of Eq. (117) vanishes, ϕ becomes the Maxwellian, and $\mathcal{L}_0$ is the usual linearized Boltzmann collision operator. Its spectrum includes a five-fold degenerate value at zero. The corresponding eigenfunctions are $\Delta \rightarrow$ linear combinations of $1, \mathbf{v}^*, v^{*2}$. These eigenfunctions are known as the summational invariants because their sum for two particles is conserved in a two-particle collision. Thus, for $\alpha = 1$ the eigenfunctions and eigenvalues of $\mathcal{L}_0$ are known and constitute the hydrodynamic modes in the long wavelength limit.

The identification of the linear combinations of these hydrodynamic modes as eigenvalues and eigenfunctions of $\mathcal{L}_0$ has been given recently [14,15] with the results

$$\mathcal{L}_0 \chi_n = \lambda_n \chi_n, \qquad (118)$$

$$\lambda_1 = 0, \quad \lambda_2 = \frac{\zeta_0}{2}, \quad \lambda_3 = \lambda_4 = \lambda_5 = -\frac{\zeta_0}{2}. \qquad (119)$$

The degeneracy for elastic collisions is partially broken, with some zero eigenvalues going to $\pm \zeta_0 / 2$. The corresponding eigenfunctions are

$$\chi_1 = 4 + v \partial_v \ln \phi(v^*), \quad \chi_2 = -3 - v \partial_v \ln \phi(v^*), \qquad (120)$$

$$\chi_n = \hat{v}_n \partial_v \ln \phi(v^*), \quad n = 3, 4, 5. \qquad (121)$$

For $\alpha = 1$, $\partial_v \ln \phi(v^*) = -2v$ and χ_n become linear combinations of $1, \mathbf{v}^*, v^{*2}$. This suggests that Eqs. (119)–(121) provide the hydrodynamic modes for $\alpha < 1$ as well. This is confirmed by noting that these eigenvalues are the same as those of the macroscopic balance equations in the long wavelength limit.

The velocity dependence of the hydrodynamic modes for $\alpha < 1$ is complicated due to their definitions in terms of the HCS distribution. An advantage of the Gaussian models is that this distribution is known explicitly and the construction of the eigenfunctions is straightforward. All of these modes are characterized by $\partial_v \ln \phi$. This has already been shown in Fig. 3 for $\alpha = 0.8$. Figure 9 shows the same data but with the result for the velocity independent collision frequency model included. The dashed curve in each case represents the elastic $\alpha = 1$ limit. For the velocity independent collision frequency χ_1 approaches a constant for large v, due to the
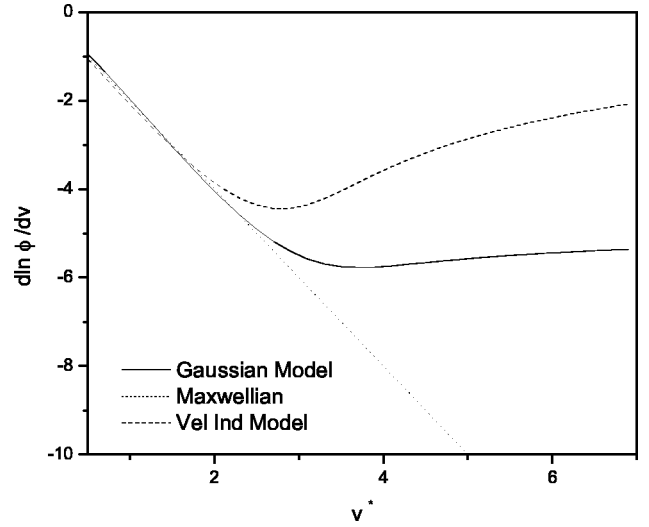

FIG. 9. Plot of $\partial_{v^*} \ln \phi(v^*)$ for the Gaussian model and velocity independent Gaussian model and the Maxwellian for $\alpha = 0.8$.

asymptotic algebraic decay of ϕ. For the velocity dependent collision frequency it approaches v according to the exponential decay of ϕ.

Figure 9 shows that there are significant qualitative differences from the hydrodynamic modes in the elastic limit when $v^* > 2$. This is the crossover of the distribution function to its large velocity form (68). The fact that the hydrodynamic modes are related to the log of the distribution function lends new importance to these asymptotic forms.

Related properties are the fluxes appearing in the Green-Kubo expressions for the transport coefficients. The expressions in the preceding section can be written in a form suggestive of Green-Kubo relations for normal fluids [33],

$$\eta = \frac{nm\ell v_0(t)}{10} \int_0^s ds' \langle D_{ij}^* \Phi_{2,ij}^*(s') \rangle e^{-1/2 \zeta^* s'}, \qquad (122)$$

$$\kappa = \frac{n\ell v_0(t)}{3} \int_0^s ds' \langle \mathbf{S}^* \cdot \boldsymbol{\Phi}_3^*(s') \rangle e^{(1/2)\zeta^* s'}, \qquad (123)$$

$$\mu = \frac{T}{n} \kappa + \frac{m\ell v_0^3(t)}{3} \int_0^s ds' \langle \mathbf{S}^* \cdot [\boldsymbol{\Phi}_1^*(s') - \boldsymbol{\Phi}_3^*(s')] \rangle. \qquad (124)$$

The brackets denote an average over the HCS in the dimensionless velocities,

$$\langle X \rangle = \int d\mathbf{v}^* \phi(\mathbf{v}^*) X(\mathbf{v}^*). \qquad (125)$$

Furthermore, the dependence on the dimensionless time s is defined by

$$X(s) = e^{s\mathcal{L}_0} X(\mathbf{V}^*). \qquad (126)$$

The averages in these expressions therefore have the interpretation of time correlation functions. The momentum flux D_{ij}^* and heat flux $\mathbf{S}^*$ are the same as in Eq. (94). They are

fluxes in the usual sense of the velocity v times linear combinations of the summational invariants $1, \mathbf{v}^*, v^{*2}$. In the elastic limit the other functions $\boldsymbol{\Phi}_n^*$ also have these forms

$$\boldsymbol{\Phi}_1^* \to 0, \quad \Phi_{2,ij}^* \to D_{ij}^*, \quad \boldsymbol{\Phi}_3^* \to \mathbf{S}^*(\mathbf{V}). \qquad (127)$$

The resulting expressions (122) and (123) for η and κ are then precisely the low density limits of the usual Green-Kubo expressions as time integrals of flux autocorrelation functions [34].

For $\alpha < 1$ the functions $\boldsymbol{\Phi}_i^*$ are no longer simply related to fluxes of the summational invariants. Instead they can be written as fluxes for the hydrodynamic modes defined above

$$\boldsymbol{\Phi}_1^* = \mathbf{v}^*(\chi_1 + \chi_2) + \tfrac{1}{2}\chi, \qquad (128)$$

$$\boldsymbol{\Phi}_{2,ij}^* = \frac{1}{2}\left(v_i \chi_j - \frac{1}{3}\delta_{ij}\mathbf{v}\cdot\chi\right), \quad j = 3,4,5, \qquad (129)$$

$$\boldsymbol{\Phi}_3^* = \tfrac{1}{2}(\mathbf{v}^*\chi_2 + \chi), \qquad (130)$$

where χ is the vector whose components are χ_n, $n = 3, 4, 5$. This relationship of the "fluxes" to the hydrodynamic modes is the same as for a normal gas. Only the forms of the hydrodynamic modes change for $\alpha < 1$. However, since these modes are significantly different at large velocities, it is expected that their effect on the transport coefficients may be important.

VII. DISCUSSION

The objective here has been to describe a simple but realistic kinetic model for the hard sphere Boltzmann equation. The new features of the Gaussian kinetic model defined in Sec. IV relative to previous models are (1) a velocity dependent collision frequency, (2) two free parameters for a good description of transport coefficients, and (3) applicability to both elastic and inelastic collisions. For elastic collisions and constant collision frequency it reduces to the ES kinetic model [17,7], while for inelastic collisions and symmetric Gaussian it reduces to the BMD model [13]. For elastic collisions, constant collision frequency, and symmetric Gaussian it becomes the usual BGK model [7]. It is also shown here that the Gaussian model for constant collision frequency can be "tuned" to represent well the more complicated Maxwell models. One motivation for the generalization of a kinetic model to include a velocity dependent collision frequency is a more accurate description of the overpopulation at large velocities for granular gases. The decrease of the distribution function for large velocities in the simplest state of HCS is algebraic for any model with a constant collision frequency, including the Maxwell model. In contrast, the decay found from the hard sphere Boltzmann equation is exponential due to the velocity dependence of the loss term in the collision operator. This qualitative difference may be important for driven states as well. Although this asymptotic behavior occurs only for extremely large velocities it can have an effect on the moments of the distribution function. In addition it has been shown in Sec. VI that the hydrodynamic modes and the Green-Kubo fluxes depend on the log of the HCS distribution function, so this asymptotic behavior is even more important. The Gaussian model with velocity dependent collision frequency incorporates this behavior and in addition gives a quite good quantitative representation of the HCS distribution function for small velocities as well. This is illustrated in Fig. 1 where $c_G(\alpha)$ shows significant improvement over the velocity independent case. As a consequence the transport coefficients $\kappa(\alpha)$ and $\mu(\alpha)$ are also significantly improved due to their dependence on $c_G(\alpha)$.

The second feature of a nonsymmetric Gaussian provides an additional parameter beyond the collision frequency that can be chosen to optimize the quality of all transport coefficients. Here they are chosen such that the shear viscosity is accurate for both the constant collision frequency and the velocity dependent collision frequency for all values of the restitution coefficient. The other transport coefficients are accurate in the elastic limit, including the correct Prandtl number for both cases. For inelastic collisions the agreement with Boltzmann remains excellent for the velocity dependent collision frequency case for all α. This is a primary improvement of the Gaussian model. In contrast, the transport coefficients from the Maxwell model are quite different from those of the hard sphere Boltzmann equation, and the other kinetic models using a symmetric Gaussian all give the wrong Prandtl number.

An advantage of most kinetic models is their structural simplicity. They can be solved exactly for many states as functionals of a few moments of the distribution. These moments still obey complicated nonlinear integral equations but the problem is simplified to the extent that exact results are often possible for states with sufficient symmetry. An example is given here for homogeneous states where the exact solution is obtained in terms of the parameters of the Gaussian gain term, $A(s)$ and $B_{ij}(s)$, which in turn are defined in terms of the moments $M_\lambda(s)$. It is shown that an arbitrary homogeneous initial condition evolves after a few collisions to a universal scaling solution, the HCS. Such behavior is expected also from the hard sphere Boltzmann equation but its complexity has precluded a proof to date. It is useful also to have the explicit representation of the HCS for other purposes as well. Here it has been noted that the hydrodynamic modes for weakly inhomogeneous states are described by eigenfunctions of the linearized Boltzmann collision operator. These eigenfunctions are determined from the HCS and a contrast with the corresponding eigenfunctions for elastic collisions was made possible by the explicit results for the HCS for the kinetic model. Significant differences are observed between the cases of the velocity independent and velocity dependent collision frequency, due to the qualitative differences in the large velocity dependences of the HCS. This is also related to the Hilbert space for formulating the eigenvalue problem for the linearized kinetic equation. The natural scalar product is an integration over the velocities weighted by the HCS distribution function. Due to the algebraic decay at large velocities for the constant collision frequency case (including the Maxwell model), polynomials of high degree do not exist in this space [14]. This restriction does not occur for the hard sphere Boltzmann equation or the Gaussian model with velocity dependent collision frequency.

It is of some interest to study any qualitative differences in the spectrum of the linearized collision operator and any consequences for the existence of hydrodynamics. The Gaussian kinetic model provides a tractable context to address this issue.

The most interesting states for experimental purposes are quasi-steady-states for systems driven at the boundaries. For states of high spatial symmetry the kinetic model again offers the advantage of an exact solution as a functional of low degree moments. An example is that of uniform shear flow where an exact solution for the distribution function has been obtained in the case of a symmetric Gaussian [35]. The result applies even for large shear rates so the rheology of states far from equilibrium can be studied directly. The Gaussian model described here also can be solved exactly for uniform shear flow and will be given elsewhere. Vibrated systems, with and without gravity, have been studied on the basis of the Boltzmann equation using Monte Carlo simulation methods leading to a number of important results bearing on experiments (e.g., boundary layers [36], dependence of velocity distribution on the distance from the driving wall [37], symmetry breaking [38]). The Gaussian kinetic model may be simple enough for a complementary analytical study of such problems.

In summary, the work here has extended earlier kinetic models to bring closer correspondence with the Boltzmann equation for the HCS and small spatial perturbations of that state. The price for these improvements is an increased complexity of the model, although this has not been an impediment for the simple states considered here. It remains to demonstrate significant new results for more complex states, not already addressed by the simpler existing models.

APPENDIX A: MOTIVATION FOR GAUSSIAN MODEL

The Gaussian model results from an approximation to the gain contribution to the Boltzmann collision operator, denoted by $g(\mathbf{r}, \mathbf{V}, t|f)$ in Eq. (30). The specific choice of a Gaussian can be interpreted as resulting from maximizing the information entropy $I[g]$,

$$I[g] = \int d\mu(\mathbf{v}) g(\mathbf{r}, \mathbf{V}, t|f) \ln g(\mathbf{r}, \mathbf{V}, t|f), \quad d\mu(\mathbf{v}) = d\mathbf{v}\nu(\mathbf{v}),$$

(A1)

among the class of functions whose weighted moments of degree 2 are specified:

$$\int d\mu(\mathbf{v}) \begin{pmatrix} 1 \\ \mathbf{v} \\ \frac{1}{2}mV^2 \end{pmatrix} g(\mathbf{r}, \mathbf{V}, t|f) = \begin{pmatrix} G_1 \\ \mathbf{G}_2 \\ G_{3ij} \end{pmatrix}.$$

(A2)

The measure for the velocity integration has been chosen to include the velocity dependence of the collision frequency. Incorporating these constraints with Lagrange multipliers and minimizing $I[g]$ leads directly to the Gaussian form

$$g(\mathbf{r}, \mathbf{V}, t|f) = \exp(-\lambda_1 - \lambda_2 \, \mathbf{v} - \lambda_{3ij} v_i v_j),$$

(A3)

where the coefficients λ_α are determined in terms of G_α from Eq. (A2). Thus, if the only known or important exact properties of the gain term are the moments in Eq. (A2) then Eq. (A3) is a "natural" choice for the model.

It may be useful to recall that $g(\mathbf{r}, \mathbf{V}, t|f)$ is exactly Gaussian for $f=$ Maxwellian at $\alpha = 1$. It has been verified numerically that this property remains true to an excellent approximation for $\alpha < 1$ as well, with only the parameters of the Gaussian changing. This gives further support for the choice (A3).

APPENDIX B: FORMAL SOLUTION FOR HOMOGENEOUS STATES

The formal solution to the Gaussian model kinetic equation (49) is

$$f^*(v^*, s) = e^{-[(1/2)\zeta^*(3 + \mathbf{v}^* \cdot \nabla_{\mathbf{v}^*}) + \nu^*(V^*)]s} f^*(v^*, 0)$$

$$+ \int_0^s ds' e^{-[(1/2)\zeta^*(3 + \mathbf{v}^* \cdot \nabla_{\mathbf{v}^*}) + \nu^*(V^*)](s - s')} \nu^*(V^*)$$

$$\times g^*(v^*, s').$$

(B1)

The action of the exponential in Eq. (B1) can be determined as follows. Define a function $X(v^*, s)$ by

$$X(v^*, s) = e^{-[(1/2)\zeta^*(3 + \mathbf{v}^* \cdot \nabla_{\mathbf{v}^*}) + \nu^*(v^*)]s} X(v^*),$$

(B2)

which then obeys the equation

$$\left(\partial_s + \frac{1}{2}\zeta^*(3 + \mathbf{v}^* \cdot \nabla_{\mathbf{v}^*}) + \nu^*(V^*)\right) X = 0.$$

(B3)

Next introduce

$$X(v^*, s) = e^{-(1/2)\zeta^* s \mathbf{v}^* \cdot \nabla_{\mathbf{v}^*}} \bar{X}(v^*, s),$$

(B4)

so that $\bar{X}(v^*, s)$ obeys the equation

$$\left(\partial_s + \frac{3}{2}\zeta^* + e^{(1/2)\zeta^* s \mathbf{v}^* \cdot \nabla_{\mathbf{v}^*}} \nu^*(v^*) e^{-(1/2)\zeta^* s \mathbf{v}^* \cdot \nabla_{\mathbf{v}^*}}\right)\bar{X} = 0.$$

(B5)

From the identity

$$e^{(1/2)\zeta^* s \mathbf{v}^* \cdot \nabla_{\mathbf{v}^*}} F(v^*) e^{-(1/2)\zeta^* s \mathbf{v}^* \cdot \nabla_{\mathbf{v}^*}} = F(e^{(1/2)\zeta^* s} v^*),$$

(B6)

this equation becomes

$$\left(\partial_s + \frac{3}{2}\zeta^* + \nu^*(e^{(1/2)\zeta^*s}v^*)\right)\bar{X} = 0. \tag{B7}$$

This can be integrated directly and inserted in Eq. (B4) to give

$$X(v^*,s) = e^{-(3/2)\zeta^*s}e^{-(1/2)\zeta^*sv^*\cdot\nabla_{v^*}}\exp\left(-\int_0^s ds'\,\nu^*(e^{(1/2)\zeta^*s'}v^*)\right)$$

$$\times X(v^*)$$

$$= e^{-(3/2)\zeta^*s}\exp\left(-\int_0^s ds'\,\nu^*(e^{-(1/2)\zeta^*s'}v^*)\right)$$

$$\times X(e^{-(1/2)\zeta^*s}v^*). \tag{B8}$$

The formal solution to the kinetic equation becomes

$$f^*(v^*,s) = e^{-(3/2)\zeta^*s}K(v^*,s)f^*(e^{-(1/2)\zeta^*s}v^*,0)$$

$$+ \int_0^s ds'\,e^{-(3/2)\zeta^*s'}K(v^*,s')\nu^*(e^{-(\zeta^*/2)s'}v^*)$$

$$\times g^*(e^{(-\zeta^*/2)s'}v^*,s-s'), \tag{B9}$$

$$K(v^*,s) = \exp\left(-\int_0^s ds'\,\nu^*(e^{-(1/2)\zeta^*s'}v^*)\right). \tag{B10}$$

It is interesting to note that no use of the explicit form for g^* has been used. So, this result applies to the Boltzmann equation as well.

APPENDIX C: MOMENT CONDITIONS

The HCS for the Gaussian model is given by Eq. (56),

$$\phi(v^*) = \int_0^\infty ds'\,e^{-(3/2)\zeta^*s'}K(v^*,s')\nu^*(e^{-(1/2)\zeta^*s'}v^*)$$

$$\times g^*(e^{-(1/2)\zeta^*s'}v^*,\infty). \tag{C1}$$

This is restricted by the moment conditions (50),

$$\begin{pmatrix} M_1^* \\ M_3^* \end{pmatrix} = \int d\mathbf{v}^*\begin{pmatrix} 1 \\ \frac{2}{3}v^{*2} \end{pmatrix}\nu^*(V^*)\phi(\mathbf{v}^*). \tag{C2}$$

These conditions can be verified by direct integration,

$$\begin{pmatrix} M_1^* \\ M_3^* \end{pmatrix} = \int d\mathbf{v}^*\begin{pmatrix} 1 \\ \frac{2}{3}v^{*2} \end{pmatrix}\nu^*(V^*)\int_0^\infty ds'\,e^{-(3/2)\zeta^*s'}K(v^*,s')$$

$$\times \nu^*(e^{-(1/2)\zeta^*s'}v^*)g^*(e^{-(1/2)\zeta^*s'}v^*,\infty)$$

$$= \int d\mathbf{v}^*\begin{pmatrix} 1 \\ \frac{2}{3}v^{*2} \end{pmatrix}\frac{1}{v^{*3}}\int_0^{v^*} dx\,x^2\nu^*(x)$$

$$\times g^*(x,\infty)\frac{2}{\zeta^*}\nu^*(V^*)\bar{K}(v^*,x),$$

with the notation

$$\bar{K}(v^*,x) \equiv \exp\left(-\frac{2}{\zeta^*}\int_x^{v^*}\frac{dx'}{x'}\nu^*(x')\right).$$

Next eliminate $2\nu^*(v^*)/\zeta^*$ by noting it can be generated by differentiating $\bar{K}$,

$$\begin{pmatrix} M_1^* \\ M_3^* \end{pmatrix} = -4\pi\int_0^\infty dv^*\begin{pmatrix} 1 \\ \frac{2}{3}v^{*2} \end{pmatrix}\int_0^{v^*} dx\,x^2\nu^*(x)$$

$$\times g^*(x,\infty)\frac{d\bar{K}(v^*,x)}{dv^*}$$

$$= -4\pi\int_0^\infty dx\,x^2\nu^*(x)g^*(x,\infty)$$

$$\times \int_x^\infty dv^*\begin{pmatrix} 1 \\ \frac{2}{3}v^{*2} \end{pmatrix}\frac{d\bar{K}(v^*,x)}{dv^*}$$

$$= 4\pi\int_0^\infty dx\,x^2\nu^*(x)g^*(x,\infty)$$

$$\times \left\{\begin{pmatrix} 1 \\ \frac{2}{3}x^2 \end{pmatrix} + \int_x^\infty dv^*\begin{pmatrix} 0 \\ \frac{4}{3}v^* \end{pmatrix}\bar{K}(v^*,x)\right\}$$

$$= \begin{pmatrix} M_1^* \\ M_3^* - \zeta^* \end{pmatrix} + 4\pi\int_0^\infty dv^*\begin{pmatrix} 0 \\ \frac{4}{3}v^* \end{pmatrix}$$

$$\times \int_0^{v^*} dx\,x^2\nu^*(x)g^*(x,\infty)\bar{K}(v^*,x), \tag{C3}$$

where use has been made of Eq. (51). The second term of Eq. (C3) can be recognized as moments of $\phi(v^*)$ to give the desired result

$$\begin{pmatrix} M_1^* \\ M_3^* \end{pmatrix} = \begin{pmatrix} M_1^* \\ M_3^* - \zeta^* \end{pmatrix} + \frac{\zeta^*}{2}\int d\mathbf{v}^*\begin{pmatrix} 0 \\ \frac{4}{3}v^{*2} \end{pmatrix}\phi(v^*) = \begin{pmatrix} M_1^* \\ M_3^* \end{pmatrix}. \tag{C4}$$

The last equality follows from Eq. (7) in the form

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \int d\mathbf{v}^*\begin{pmatrix} 1 \\ \frac{2}{3}v^{*2} \end{pmatrix}\phi(\mathbf{v}^*). \tag{C5}$$

This confirms the consistency of moment conditions (50) and (51).

The two equations (C5) fix the values of A^* and B^{*-1} in the Gaussian model (49). A convenient representation is

$$\frac{3}{2} = \frac{\int_0^\infty dv^* v^* \int_0^{v^*} dx x^2 \nu^*(x) e^{-B^{*-1}x^2} \bar{K}(v^*,x)}{\int_0^\infty dv^* v^{*-1} \int_0^{v^*} dx x^2 \nu^*(x) e^{-B^{*-1}x^2} \bar{K}(v^*,x)},$$

or

$$0 = \int_0^\infty dv^* \left(v^* - \frac{3}{2}v^{*-1}\right) \int_0^{v^*} dx x^2 \nu^*(x) e^{-B^{*-1}x^2} \bar{K}(v^*,x). \tag{C6}$$

This determines B^*. Next A^* is obtained from

$$A^{*-1} = \frac{8\pi}{\zeta^*} \int_0^\infty dv^* v^{*-1} \int_0^{v^*} dx x^2 \nu^*(x) e^{-B^{*-1}x^2} \bar{K}(v^*,x). \tag{C7}$$

Finally, with A^* and B^{*-1} known, the moments are determined from Eq. (51),

$$\begin{pmatrix} M_1^* \\ M_3^* - \zeta^* \end{pmatrix} = \int d\mathbf{v}^* \begin{pmatrix} 1 \\ \frac{2}{3}v^{*2} \end{pmatrix} \nu^*(V^*) A^* e^{-B^{*-1}v^{*2}}. \tag{C8}$$

APPENDIX D: LINEARIZED COLLISION OPERATOR

The collision operator for the Gaussian model is

$$J(\mathbf{r},\mathbf{v}|f(t)) \equiv -\nu(\mathbf{r},V,t)[f(\mathbf{r},\mathbf{V},t) - g(\mathbf{r},\mathbf{V},t|f)]. \tag{D1}$$

The distribution function is expanded as

$$f(\mathbf{r},\mathbf{V},t) = f^{(0)}(\mathbf{r},\mathbf{V},t) + f^{(1)}(\mathbf{r},\mathbf{V},t) + \cdots, \tag{D2}$$

$$g(\mathbf{r},\mathbf{V},t|f) = g(\mathbf{r},\mathbf{V},t|f^{(0)}) + \int d\mathbf{v}' \left. \frac{\delta g(\mathbf{r},\mathbf{V},t|f)}{\delta f(\mathbf{r},\mathbf{v}',t)} \right|_{f=f^{(0)}} \times f^{(1)}(\mathbf{r},\mathbf{V}',t) + \cdots. \tag{D3}$$

The linearized collision operator is therefore

$$\mathcal{L}f^{(1)} = -\int d\mathbf{v}' \left. \frac{\delta J(\mathbf{r},\mathbf{v}|f(t))}{\delta f(\mathbf{r},\mathbf{v}',t)} \right|_{f=f^{(0)}} f^{(1)}(\mathbf{r},\mathbf{V}',t)$$

$$= \nu(\mathbf{r},V,t) \Bigg[f^{(1)}(\mathbf{r},\mathbf{V},t) - \int d\mathbf{v}' \left. \frac{\delta g(\mathbf{r},\mathbf{V},t|f)}{\delta f(\mathbf{r},\mathbf{v}',t)} \right|_{f=f^{(0)}} f^{(1)}(\mathbf{r},\mathbf{V}',t) \Bigg]. \tag{D4}$$

The second term of Eq. (D3) can be made more explicit by recalling that the functional dependence of $g(\mathbf{r},\mathbf{V},t|f)$ occurs only through A and B_{ij}:

$$\int d\mathbf{v}' \left. \frac{\delta g(\mathbf{r},\mathbf{V},t|f)}{\delta f(\mathbf{r},\mathbf{v}',t)} \right|_{f=f^{(0)}} f^{(1)}(\mathbf{r},\mathbf{V}',t)$$

$$= g(\mathbf{r},\mathbf{V},t|f^{(0)}) \Bigg[\int d\mathbf{v}' \left. \frac{\delta \ln A(\mathbf{r},t|f)}{\delta f(\mathbf{r},\mathbf{v}',t)} \right|_{f=f^{(0)}} f^{(1)}$$

$$\times (\mathbf{r},\mathbf{V}',t) - V_i V_j \int d\mathbf{v}' \left. \frac{\delta B_{ij}^{-1}(\mathbf{r},t|f)}{\delta f(\mathbf{r},\mathbf{v}',t)} \right|_{f=f^{(0)}}$$

$$\times f^{(1)}(\mathbf{r},\mathbf{V}',t) \Bigg]. \tag{D5}$$

Using the definition of B_{ij} in Eq. (39) gives

$$\frac{\delta B_{ij}^{-1}(\mathbf{r},t|f)}{\delta f(\mathbf{r},\mathbf{v}',t)} = \left(-B^{-1}(\mathbf{r},t|f^{(0)}) \frac{\delta B(\mathbf{r},t|f)}{\delta f(\mathbf{r},\mathbf{v}',t)} B^{-1}(\mathbf{r},t|f^{(0)}) \right)_{ij}$$

$$= -\frac{9}{B_{kk}^2} \left(\frac{1}{3} \frac{\delta B_{kk}(\mathbf{r},t|f)}{\delta f(\mathbf{r},\mathbf{v}',t)} \delta_{ij} + \frac{y(\alpha)}{nm} D_{ij}(\mathbf{V}) \right). \tag{D6}$$

Then Eq. (D5) becomes

$$\int d\mathbf{v}' \left. \frac{\delta g(\mathbf{r},\mathbf{V},t|f)}{\delta f(\mathbf{r},\mathbf{v}',t)} \right|_{f=f^{(0)}} f^{(1)}(\mathbf{r},\mathbf{V}',t)$$

$$= g(\mathbf{r},\mathbf{V},t|f^{(0)}) \Bigg[\int d\mathbf{v}' \left. \frac{\delta \ln A(\mathbf{r},t|f)}{\delta f(\mathbf{r},\mathbf{v}',t)} \right|_{f=f^{(0)}} f^{(1)}$$

$$\times (\mathbf{r},\mathbf{V}',t) + V^2 \frac{3}{B_{kk}^2} \int d\mathbf{v}' \left. \frac{\delta B_{kk}(\mathbf{r},t|f)}{\delta f(\mathbf{r},\mathbf{v}',t)} \right|_{f=f^{(0)}} f^{(1)}$$

$$\times (\mathbf{r},\mathbf{V}',t) + \frac{9y(\alpha)}{nm^2 B_{kk}^2} D_{ij}(\mathbf{V})$$

$$\times \int d\mathbf{v}' D_{ij}(\mathbf{V}') f^{(1)}(\mathbf{r},\mathbf{V}',t) \Bigg]. \tag{D7}$$

The expansion (D2) leads to a corresponding expansion for the moments

$$\begin{pmatrix} M_1 \\ \mathbf{M}_2 \\ M_3 \end{pmatrix} = \int d\mathbf{v} \begin{pmatrix} 1 \\ \mathbf{v} \\ \frac{1}{2}m(\mathbf{v}-\mathbf{u})^2 \end{pmatrix}$$

$$\times \nu(\mathbf{r},V,t)[f^{(0)}(\mathbf{r},\mathbf{V},t) + f^{(1)}(\mathbf{r},\mathbf{V},t) + \cdots]$$

$$= \begin{pmatrix} M_1^{(0)} \\ \mathbf{M}_2^{(0)} \\ M_3^{(0)} \end{pmatrix} + \begin{pmatrix} M_1^{(1)} \\ \mathbf{M}_2^{(1)} \\ M_3^{(1)} \end{pmatrix} + \cdots. \tag{D8}$$

The coefficients $A(\mathbf{r},t|f^{(0)})$ and $B_{ij}(\mathbf{r},t|f^0) = [B_{kk}(\mathbf{r},t|f^0)/3]\delta_{ij} = B\delta_{ij}$ are determined from $M_\lambda^{(0)}$ just as is done in Appendix C. The remaining part of the moment conditions are

$$\begin{pmatrix} M_1^{(1)} \\ \mathbf{M}_2^{(1)} \\ M_3^{(1)} \end{pmatrix} = \int d\mathbf{v} \begin{pmatrix} 1 \\ \mathbf{v} \\ \frac{1}{2}mV^2 \end{pmatrix} \nu(\mathbf{r},V,t)$$

$$\times \int d\mathbf{v}' \left. \frac{\delta g(\mathbf{r},\mathbf{V},t|f)}{\delta f(\mathbf{r},\mathbf{v}',t)} \right|_{f=f^{(0)}} f^{(1)}(\mathbf{r},\mathbf{V}',t).$$

(D9)

The terms on the right side of Eq. (D7) can be identified as an expansion in terms of polynomials of degree 2 in the velocity. To do so, first define a Hilbert space with scalar product

$$(a,b) = \int d\mathbf{v} \, \nu g^{(0)} a^* b, \quad g^{(0)} = g(|f^{(0)}). \quad \text{(D10)}$$

Next, define the set of functions $\{\psi_\sigma\}$,

$$\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{1}{N_1}} 1 \\ \sqrt{\frac{3}{N_2}} \mathbf{V} \\ \sqrt{\frac{1}{N_3}} \left(V^2 - \frac{N_2}{N_1} \right) \end{pmatrix}, \quad \text{(D11)}$$

with normalization constants

$$N_1 = (1,1), \quad N_2 = (V_i, V_i) = (1,V^2),$$

$$N_3 = \left(\left(V^2 - \frac{N_2}{N_1} \right), \left(V^2 - \frac{N_2}{N_1} \right) \right) = (1,V^4) - \frac{N_2^2}{N_1}. \quad \text{(D12)}$$

These functions form an orthonormal set

$$(\psi_\sigma, \psi_\mu) = \delta_{\sigma\mu}. \quad \text{(D13)}$$

Equation (D7) may now be written in the form

$$\int d\mathbf{v}' \left. \frac{\delta g(\mathbf{r},\mathbf{V},t|f)}{\delta f(\mathbf{r},\mathbf{v}',t)} \right|_{f=f^{(0)}} f^{(1)}(\mathbf{r},\mathbf{V}',t)$$

$$= g(\mathbf{r},\mathbf{V},t|f^{(0)}) \left[e_\sigma \psi_\alpha(\mathbf{V}) + \frac{y(\alpha)}{nm^2B^2} D_{ij}(\mathbf{V}) \right.$$

$$\left. \times \int d\mathbf{v}' D_{ij} f^{(1)} \right]. \quad \text{(D14)}$$

The coefficients e_σ can be determined by taking the scalar product of this equation with ψ_μ:

$$e_\mu = \int d\mathbf{v} \, \nu \psi_\mu \int d\mathbf{v}' \left. \frac{\delta g(\mathbf{r},\mathbf{V},t|f)}{\delta f(\mathbf{r},\mathbf{v}',t)} \right|_{f=f^{(0)}} f^{(1)}(\mathbf{r},\mathbf{V}',t)$$

$$= \int d\mathbf{v} \, \nu \psi_\mu f^{(1)}. \quad \text{(D15)}$$

The second equality follows from Eq. (D9) and allows these terms to be represented as a projection onto the subspace spanned by $\{\psi_\sigma\}$:

$$\nu(\mathbf{r},V,t) \int d\mathbf{v}' \left. \frac{\delta g(\mathbf{r},\mathbf{V},t|f)}{\delta f(\mathbf{r},\mathbf{v}',t)} \right|_{f=f^{(0)}} f^{(1)}(\mathbf{r},\mathbf{V}',t)$$

$$= \mathcal{P}\nu f^{(1)} + \frac{y}{nm^2B^2} \nu g^{(0)} D_{ij} \int d\mathbf{v}' D_{ij} f^{(1)}, \quad \text{(D16)}$$

where $\mathcal{P}$ is a projection operator,

$$\mathcal{P}X = \nu g^{(0)} \psi_\sigma \int d\mathbf{v} \psi_\sigma X. \quad \text{(D17)}$$

The linearized collision operator of Eq. (D4) now takes the simple form

$$\mathcal{L}f^{(1)} = (1-\mathcal{P})\nu f^{(1)} - \frac{y}{nm^2B^2} \nu g^{(0)} D_{ij}(\mathbf{V}) \int d\mathbf{v}' D_{ij} f^{(1)}. \quad \text{(D18)}$$

The first term represents the fact that $\mathcal{L}$ has a null subspace due to the moment conditions

$$\int d\mathbf{v} \begin{pmatrix} 1 \\ \mathbf{V} \\ \frac{1}{2}mV^2 \end{pmatrix} \mathcal{L}f^{(1)} = 0. \quad \text{(D19)}$$

This is the usual BGK-like operator with a single, infinitely degenerate point in the spectrum for all functions of the orthogonal subspace. The second term is a projection onto a specific function in the orthogonal subspace and is the new effect of the asymmetric Gaussian approximation, or the nonzero value of $\tilde{B}_{ij}$.

Finally, noting that

$$\int d\mathbf{v} D_{ij} D_{ij} g^{(0)} = \frac{5}{2} nm^2B^2, \quad \text{(D20)}$$

allows the linearized operator to be written as

$$\mathcal{L}f^{(1)} = (1-\mathcal{P})\nu f^{(1)} - \frac{5}{2}y \frac{\nu g^{(0)} D_{ij}(\mathbf{V}) \int d\mathbf{v}' D_{ij} f^{(1)}}{\int d\mathbf{v} D_{ij} D_{ij} g^{(0)}}. \quad \text{(D21)}$$

[1] J. J. Brey, J. W. Dufty, and A. Santos, J. Stat. Phys. **87**, 1051 (1997).

[2] M. H. Ernst, in *Dynamics: Models and Kinetic Methods for Non-equilibrium Many Body Systems*, edited by J. Karkheck (Kluwer Academic, Boston, 2000).

[3] T. P. C van Noije and M. H. Ernst, in *Granular Gases*, edited by T. Pöschel and S. Luding, Lecture Notes in Physics Vol. 564 (Springer, Berlin, 2001), pp. 3–30. J. J. Brey and D. Cubero, *Granular Gases*, pp. 59–78.

[4] A. Goldshtein and M. Shapiro, J. Fluid Mech. **282**, 41 (1995).

[5] For a recent formal derivation see J. W. Dufty, Adv. Complex Syst. **4**, 397 (2001).

[6] G. A. Bird, *Molecular Gas Dynamics and the Direct Simulation of Gas Flows* (Clarendon, Oxford, 1994).

[7] C. Cercignani, *Theory and Application of the Boltzmann Equation* (Elsevier, New York, 1975).

[8] J. W. Dufty in *Lectures on Thermodynamics and Statistical Mechanics*, edited by M. L'opez de Haro and C. Varea (World Scientific, River Edge, NJ, 1990), Vol. 166.

[9] V. Garzó and A. Santos, *Kinetic Theory of Gases in Shear Flows. Nonlinear Transport* (Kluwer Academic, Dordrecht, 2003).

[10] J. Javier Brey, in *Dynamics: Models and Kinetic Methods for Non-equilibrium Many Body Systems* (Ref. [2]).

[11] J. Javier Brey, M. J. Ruiz-Montero, and F. Moreno, Phys. Rev. E **55**, 2846 (1997).

[12] M. Tij, E. Tahiri, J. M. Montanero, V. Garzó, A. Santos, and J. Dufty, J. Stat. Phys. **103**, 1035 (2001).

[13] J. J. Brey, F. Moreno, and J. W. Dufty, Phys. Rev. E **54**, 445 (1996).

[14] J. W. Dufty and J. J. Brey, Phys. Rev. E **68**, 030302 (2003).

[15] J. Javier Brey, James W. Dufty, and Maria Jose Ruiz-Montero, in *Granular Gases*, edited by T. Poeschel (Springer, Berlin, 2003); e-print cond-mat/0302180.

[16] For a review with references see E. Ben-Naim and P. L. Krapivsky, e-print cond-mat/0301238.

[17] L. H. Holway, in *Rarefied Gas Dynamics* (Academic Press, New York, 1966), Vol. I, p. 193.

[18] T. P. C van Noije and M. H. Ernst, Granular Matter **1**, 57 (1998).

[19] J. Javier Brey, M. J. Ruiz-Montero, and D. Cubero, Phys. Rev. E **54**, 3664 (1996).

[20] J. M. Montanero and A. Santos, Granular Matter **2**, 53 (2000).

[21] J. J. Brey, J. W. Dufty, C. S. Kim, and A. Santos, Phys. Rev. E **58**, 4638 (1998).

[22] J. Javier Brey, D. Cubero, and M. J. Ruiz-Montero, Phys. Rev. E **59**, 1256 (1999).

[23] A. Santos, J. M. Montenero, J. W. Dufty, and J. J Brey, Phys. Rev. E **57**, 1664 (1998); J. W. Dufty, A. Santos, and J. Brey, Phys. Rev. Lett. **77**, 1270 (1996).

[24] J. F. Lutsko, Phys. Rev. Lett. **78**, 243 (1997); Phys. Rev. E **58**, 434 (1998).

[25] See, for example, E. Ben-Naim and P. L. Krapivsky, Phys. Rev. E **61**, R5 (2000); M. Ernst and R. Brito, *Granular Gas Dynamics*, in Lecture Notes in Physics Vol. 624 (Springer, Berlin, 2003).

[26] A. Santos, Physica A **321**, 442 (2003).

[27] P. L. Krapivsky and E. Ben-Naim, J. Phys. A **35**, L147 (2002).

[28] M. Ernst and R. Brito, Europhys. Lett. **58**, 182 (2002).

[29] J. Brey, J. Dufty, and A. Santos, J. Stat. Phys. **97**, 281 (1999).

[30] A. Astillero and A. Santos, e-print cond-mat/0309220.

[31] Velocity dependent models for elastic collisions are described in Ref. [7]; see also J. Javier Brey and A. Santos, J. Stat. Phys. **67**, 123 (1984); V. Garzó, Mol. Phys. **63**, 517 (1988).

[32] For the case of a velocity independent collision frequency and elastic collisions the Gaussian model reduces to the ES model of Ref. [17] . For inelastic collisions it constitutes a generalization of Ref. [13] to the nonspherical Gaussian model.

[33] J. W. Dufty and J. J. Brey, J. Stat. Phys. **109**, 433 (2002).

[34] J. A. McLennan, *Introduction to Nonequilibrium Statistical Mechanics* (Prentice-Hall, Englewood Cliffs, NJ, 1989).

[35] J. J. Brey, M. J. Ruiz-Montero, and F. Moreno, Phys. Rev. E **55**, 2846 (1997).

[36] J. Javier Brey, M. J. Ruiz-Montero, and F. Moreno, Phys. Rev. E **62**, 5339 (2000); **63**, 061305 (2001).

[37] J. Javier Brey and M. J. Ruiz-Montero, Phys. Rev. E **67**, 021307 (2003).

[38] J. Javier Brey, M. J. Ruiz-Montero, F. Moreno, and R. Garcia-Rojo, Phys. Rev. E **65**, 061302 (2002).